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COERCIVE SINGULARLY PERTURBED WIENER-HOPF OPERATORS AND APPLICATIONS

W. D. WENDT

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PROMOTOR: PROF.DR. L.S. FRANK

COREFERENTEN: PROF.DR.J.J. DUISTERMAAT

PROF.DR. Y. KANNAI

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PROEFSCHRIFT

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I. INTRODUCTION

The Dirichlet problem for singularly perturbed strongly elliptic differential equations with homogeneous boundary conditions has first been systematically investigated in [V-L], where asymptotic formulae for the solutions in the case of C^∞ data were indicated and sharp error estimates were proved. These estimates, as well as the existence results in [V-L], were obtained by using Gårding type inequalities for C^∞-functions with compact support. In [E], the Dirichlet problem with homogeneous boundary conditions was investigated for some classes of singularly perturbed pseudodifferential operators. In [D], some singularly perturbed boundary value problems with complementary potentials were considered, too (see [V-E] for the treatment of this kind of problems without small parameter).

In [Fr 1], the Sobolev type spaces with vectorial index were introduced, which are adequate to the treatment of elliptic singular perturbation problems with general boundary conditions, the necessary and sufficient condition for the stability (coerciveness condition) was stated and a two-sided a-priori-estimate was proved, which holds uniformly with respect to the small parameter ε. These results were obtained by an extension of the classical parametrix construction and by accurate estimates of the corresponding Poisson kernels (see also [Fi], where some sufficient conditions for the stability were indicated and one-sided estimates of Schauder type were established for operators with constant coefficients in a half space).

A different technique, namely a suitable factorization method for the operator associated with singularly perturbed coercive differential boundary value problems, was developed later in [Fr-W]. This technique is based upon the extension of the algebra of Wiener-Hopf operators introduced in [B.d.M] to an operator algebra with small parameter. It enables one to reduce a coercive singular perturbation to a regular one and to obtain new asymptotic formulae for their solutions in the differential case. Singularly perturbed ordinary differential equations with general boundary conditions had been previously treated in [Fr 2], using a different method, based upon the construction of a quasi-inverse operator.

The main motivation for the extension of the investigation to a more general algebra of Wiener-Hopf operators with small parameter was the fact that this kind of operators appears in a natural way while constructing parametrices and reducing operators for coercive singularly perturbed differential boundary value problems. Also the fact that such operators appear in applications (see, for instance, [D]), was a stimulating factor for such an extension.

In this thesis, the stability theory for coercive differential singular perturbations is extended to some classes of Wiener-Hopf operators with small parameter. Operators which have the form of columns where all the symbols are rational functions of the conormal variable, are considered. The latter restriction - which can be weakened - is motivated by the fact that all the Wiener-Hopf operators which arise in the reduction of singularly perturbed coercive differential boundary value problems to regular perturbations, have this property.

To this kind of operators correspond singularly perturbed pseudodifferential equations in a bounded domain with general boundary conditions, where all the symbols are rational functions of the conormal variable, and where the index for both the perturbed and for the reduced symbol is nonnegative. Systems of pseudodifferential equations with general boundary conditions and general index, however without small parameter have been investigated in [V-E], where the coerciveness condition was formulated and its equivalency with a two-sided a-priori-estimate was proved. In the investigation of Wiener-Hopf operators with small parameter, it turns out that the smoothness condition from [V-E] is, in general, not satisfied uniformly with respect to ε. There is a need to introduce a new class of symbols, called here $\mathcal{D}^{k_0}$, the elements of which are smooth in the Sobolev space H_s uniformly with respect to ε, if $s < k_0+\frac{1}{2}$.

The reduced problem is characterized by the presence of l_0 boundary conditions, where the number l_0 is, in general, not equal to the index of the reduced pseudodifferential symbol (as in the differential case), but is determined by a condition which involves the orders of the boundary operators.

The statement of the coerciveness condition for the differential case ([Fr 1]), can be extended to the Wiener-Hopf operators considered here. Indeed, one has to replace the condition of linear independence of certain polynomials by the condition that the corresponding determinants do not vanish (as it was done in [V-E] for problems without small parameter).

In the case of Wiener-Hopf column operators with small parameter, the formulation of the coerciveness condition is the same as in section 3.3 of [Fr 1], with the difference, that the number l_0 appears instead of the index of the reduced symbol.

The proof of the two-sided a priori estimate, for which the coerciveness condition is necessary and sufficient, is based on a combination of the methods from [Fr 1],[V-E] and of a factorization technique which can be applied to symbols of the form $a(\varepsilon\xi) \in \mathcal{D}^{k_0}$.

As in the differential case ([Fr-W]), one can reduce a singularly perturbed coercive Wiener-Hopf equation to a regular perturbation, using the parametrix constructed in the proof of the stability result. With $\mathfrak{A}^\varepsilon, \mathfrak{A}^0$ being the operators corresponding to the perturbed and reduced problems, respectively, operators R_l^ε, R_r^ε (the so-called left and right reducing operators) are constructed, such that for some $\gamma > 0$,

$$\mathfrak{A}^\varepsilon = R_l^\varepsilon \circ \mathfrak{A}^0 \circ R_r^\varepsilon + O(\varepsilon^\gamma) \quad \varepsilon \downarrow 0. \tag{1.1}$$

The main difference with [Fr-W] lies in the fact that in the differential case, one can take for R_r^ε the identity operator, whereas in the case considered here, in general, R_r^ε must be chosen in a different way for (1.1) to hold. Analogously to [Fr-W], quasi-inverse operators S_l^ε, S_r^ε to R_l^ε, R_r^ε are constructed, i.e. operators which satisfy the conditions

$$R_l^\varepsilon S_l^\varepsilon = \mathrm{Id} + O(\varepsilon^\gamma), \; S_l^\varepsilon R_l^\varepsilon = \mathrm{Id} + O(\varepsilon^\gamma)$$

$$R_r^\varepsilon S_r^\varepsilon = \mathrm{Id} + O(\varepsilon^\gamma), \; S_r^\varepsilon R_r^\varepsilon = \mathrm{Id} + O(\varepsilon^\gamma)$$

for $\varepsilon \downarrow 0$. If the reduced problem has a unique solution, one gets the conclusion that $\mathfrak{A}^\varepsilon$ is invertible for $\varepsilon \in (0,\varepsilon_0]$ with $0 < \varepsilon_0 \ll 1$. It turns out that $(\mathfrak{A}^\varepsilon)^{-1}$ can for these ε be expanded in a convergent series and can be expressed in terms of the operators $(\mathfrak{A}^0)^{-1}$, S_l^ε, S_r^ε. In the case when the reduced problem does not have a unique solution, (1.1) guarantees the stability of the index under coercive singular perturbations (as in [Fr-W]).

In the case of the Dirichlet boundary value problem with homogeneous boundary conditions for singularly perturbed elliptic pseudodifferential operators, the reduction procedure does not require the use of the Boutet de Monvel's type algebra of operators with small parameter and relies on the Wiener-Hopf factorization of the corresponding symbols (see [E]).

In order to be more specific, several examples will be discussed. Let $U \subset \mathbb{R}^n$ be a bounded domain, let $F_{x\to\xi}$ and $F^{-1}_{\xi\to x}$ be the direct and inverse Fourier transforms, respectively, and let

$$\mathrm{Op}\ a(x,\varepsilon,\xi)u = F^{-1}_{\xi\to x}\, a(x,\varepsilon,\xi)F_{x\to\xi}u$$

be the pseudodifferential operator with the symbol $a(x,\varepsilon,\xi)$. With π and l_0 being the restriction operator to U and the extension operator by zero, respectively, consider the Wiener-Hopf equation

$$\text{(1.2)} \qquad \pi\ \mathrm{Op}\ L_0(\varepsilon\ \xi)\ l_0u(x) = f(x), \qquad x \in U$$

with $L_0(\eta) = (1+2|\eta|^2)^{-1}(1+|\eta|^2)$.

The problem (1.2) satisfies the coerciveness condition, which in this case is the ellipticity condition from [Fr 1]. Namely, one has

$$C^{-1} \le |L_0(\varepsilon\ \xi)| \le C \qquad \forall\varepsilon \in \mathbb{R}_+,\ \forall\xi \in \mathbb{R}^n$$

with a constant C independent upon ε,ξ. Moreover, the symbol L_0 belongs to the class $\mathcal{D}^{k_0}$ with $k_0 = 0$. As in [Fr 1], let $H_{(s)}$, $s \in \mathbb{R}^3$, be the Sobolev type space of all functions $u : (0,\varepsilon_0] \times \mathbb{R}^n \to \mathbb{C}$ for which $\sup_{\varepsilon\in(0,\varepsilon_0]} ||u||_{(s)}$ is finite, where

$$||u||_{(s)} = ||\varepsilon^{-s_1}(1-\Delta)^{s_2/2}(1-\varepsilon^2\Delta)^{s_3/2}u||_{L^2(\mathbb{R}^n)}.$$

The following two-sided a priori estimate holds for the solution of (1.2) with $-\frac{1}{2} < s_2 < k_0+\frac{1}{2}$, $-\frac{1}{2} < s_2+s_3$:

$$\text{(1.3)} \qquad C^{-1}||f||_{(s)} \le ||u||_{(s)} \le C||f||_{(s)} \qquad \forall f \in H_{(s)}(U),$$

where $||\ ||_{(s)}$ denotes the norm in the space $H_{(s)}(U)$ and where C does not depend upon ε,f,u. For $s_2 > k_0+\frac{1}{2}$, both inequalities in (1.3) fail to hold. Indeed, if $u \in C^\infty(\bar{U})$ uniformly with respect to ε, the right hand side f of

(1.2) is not uniformly bounded with respect to ε in the norm $\| \|_{(s)}$ because the kernel of the integral operator π Op $L_0 1_0$ contains boundary layer type functions. For $s_2+s_3 < -\frac{1}{2}$, (1.3) does not even hold with C dependent upon ε (see [V-E]).

For the Wiener-Hopf equation

$$(1.4) \qquad \pi \text{ Op } \frac{1-\varepsilon^2|\xi|^2}{1+2\varepsilon^2|\xi|^2} 1_0 u(x) = f(x), \qquad x \in U,$$

the two-sided estimate (1.3) does not hold uniformly with respect to ε. Indeed, following [Fr 1], let u(x) be defined by

$$u(x) = e^{i\varepsilon^{-1} x\eta} \psi(x),$$

where $\eta \in \mathbb{R}^n$, $|\eta| = 1$, is a fixed vector and where $\psi \in C_0^\infty(U)$. The norm $\|u\|_{(s)}$ is of order $\varepsilon^{-s_1-s_2}$ for $\varepsilon \downarrow 0$, whereas $\|f\|_{(s)}$ is of order $\varepsilon^{1-s_1-s_2}$, due to the fact that the exponential factor in u(x) is a solution to the homogeneous equation $\mathrm{Op}(1-\varepsilon^2|\xi|^2)v \equiv 0$. The reason for this instability is that the pseudodifferential operator in (1.4) does not satisfy the ellipticity condition in [Fr 1].

With (a_{kj}) a positive definite matrix, consider the boundary value problem

$$(1.5) \qquad \pi \text{ Op} \frac{(1+\Sigma a_{kj}\xi_k\xi_j)(1+\varepsilon^2|\xi|^2)}{(2+|\xi|^2)} 1_0 u(x) = f(x), \quad x \in U$$

$$(1.6) \qquad \pi_0 \text{ Op } b(x',\varepsilon,\xi) 1_0 u(x') = \phi(x'), \quad x' \in \partial U,$$

where π_0 denotes the restriction operator to ∂U and where Op b is a pseudodifferential operator of vectorial order $\mu = (0,m,p)$. With (1.5), (1.6) is associated the Wiener-Hopf operator

$$\mathcal{A}^\varepsilon = \begin{pmatrix} \pi \text{ Op} \dfrac{(1+\Sigma a_{kj}\xi_k\xi_j)(1+\varepsilon^2|\xi|^2)}{(2+|\xi|^2)} 1_0 \\ \pi_0 \text{ Op } b\ 1_0 \end{pmatrix}$$

For $m > -1$, the solution u is sought in the space $H_{(s)}(U)$ with $-\frac{1}{2} < s_2 < m+\frac{1}{2}$, $\max(-1,m+p)+\frac{1}{2} < s_2+s_3$ and the reduced problem of (1.5), (1.6) is given by

$$(1.7) \qquad \pi \ \mathrm{Op} \ \frac{1+\Sigma a_{kj}\xi_k\xi_j}{2+|\xi|^2} \ 1_0 u(x) = f(x), \quad x \in U,$$

where the solution belongs to the space $H_{s_2}(U)$. As in [Fr 1], the condition $s_2 < m+\frac{1}{2}$ appears due to the fact that the boundary layer contribution to the solution is not uniformly bounded with respect to ε in the norm of $H_{(s)}(U)$ with $s_2 > m+\frac{1}{2}$. For $s_2 < -\frac{1}{2}$, the reduced operator $\pi \ \mathrm{Op} \ \frac{1+\Sigma a_{kj}\xi_k\xi_j}{2+|\xi|^2} 1_0$ is not bounded from $H_{(s)}(U)$ into $H_{(s)}(U)$.

On the other hand, for $m < -1$, the solution of (1.5), (1.6) is sought in the space $\mathring{H}_{(s)}(U)$ of functions in $H_{(s)}(\mathbb{R}^n)$ which have their support in $\overline{U}$, where $-\frac{3}{2} < s_2 < -\frac{1}{2}$, $\max(-1,m+p)+\frac{1}{2} < s_2+s_3 < \frac{1}{2}$. The reduced problem is in this case given by

$$(1.8) \qquad \pi \ \mathrm{Op} \ \frac{1+\Sigma a_{kj}\xi_k\xi_j}{2+|\xi|^2} \ u(x) = f(x), \quad x \in U$$

$$(1.9) \qquad \pi_0 \ \mathrm{Op} \ b(x',0,\xi)u(x') = \phi(x'), \quad x' \in \partial U$$

where $u \in \mathring{H}_{s_2}(U)$ is sought. The conditions $|s_2+s_3| < \frac{1}{2}$ and $|s_2+1| < \frac{1}{2}$ appear because otherwise the perturbed and reduced problems cannot be well-posed. Thus, for the problem (1.5), (1.6), the formulations of the reduced problem and of the coerciveness condition depend upon the order of the boundary operator Op b.

Following [Fr-W], the reduction of coercive singular perturbations to regular ones will be illustrated. Consider the following singularly perturbed pseudodifferential equation:

$$(1.10) \qquad \mathrm{Op} \ L(\varepsilon,\xi)u(x) = f(x), \qquad x \in \mathbb{R}^n$$

where

$$L(\varepsilon,\xi) = (2+|\xi|^2)^{-1}(1+\Sigma a_{kj}\xi_k\xi_j)(1+\varepsilon^2|\xi|^2).$$

Since

$$C^{-1}(1+\varepsilon^2|\xi|^2) \le |L(\varepsilon,\xi)| \le C(1+\varepsilon^2|\xi|^2)$$

with C independent upon $\varepsilon \in \mathbb{R}_+$, $\xi \in \mathbb{R}^n$, Op L is a bounded operator from $H_{(0,0,2)}(\mathbb{R}^n)$ to $H_{(0,0,0)}(\mathbb{R}^n)$:

$$\mathrm{Op}\ L(\varepsilon,\xi) \in \mathrm{Hom}(H_{(0,0,2)}(\mathbb{R}^n),\ H_{(0,0,0)}(\mathbb{R}^n)).$$

The so-called principal symbol of L is given by

$$L_0(\varepsilon,\xi) = |\xi|^{-2}(\Sigma a_{kj}\xi_k\xi_j)(1+\varepsilon^2|\xi|^2)$$

Let the symbols $R_1(\varepsilon,\xi)$, $S_1(\varepsilon,\xi)$ be defined in terms of $L_0(\varepsilon,\xi)$ by

$$R_1(\varepsilon,\xi) = L_0(\varepsilon,\hat{\xi})(L_0(0,\hat{\xi}))^{-1} \tag{1.11}$$

$$S_1(\varepsilon,\xi) = L_0(0,\hat{\xi})(L_0(\varepsilon,\hat{\xi}))^{-1} \tag{1.12}$$

where $\hat{\xi} = (1+|\xi|^2)^{\frac{1}{2}}|\xi|^{-1}\xi$. These symbols satisfy the inequalities

$$C^{-1}(1+\varepsilon^2|\xi|^2) \le |R_1(\varepsilon,\xi)| \le C(1+\varepsilon^2|\xi|^2) \tag{1.13}$$

$$C^{-1}(1+\varepsilon^2|\xi|^2)^{-1} \le |S_1(\varepsilon,\xi)| \le C(1+\varepsilon^2|\xi|^2)^{-1} \tag{1.14}$$

with C independent upon $\varepsilon \in \mathbb{R}_+$, $\xi \in \mathbb{R}^n$.

The operator $\mathrm{Op}\ L(\varepsilon,\xi)$ can be factorized up to a small operator:

$$\mathrm{Op}\ L(\varepsilon,\xi) = \mathrm{Op}\ R_1(\varepsilon,\xi)\ \mathrm{Op}\ L(0,\xi)+\varepsilon Q \tag{1.15}$$

with $Q \in \mathrm{Hom}(H_{(0,0,2)}, H_{(0,0,0)})$.

Moreover, Op R_1 and Op S_1 are quasi-invers to each other:

$$\begin{aligned} &\mathrm{Op}\ R_1(\varepsilon,\xi)\ \mathrm{Op}\ S_1(\varepsilon,\xi) = \mathrm{Id} + \varepsilon Q_1 \\ &\mathrm{Op}\ S_1(\varepsilon,\xi)\ \mathrm{Op}\ R_1(\varepsilon,\xi) = \mathrm{Id} + \varepsilon Q_2 \end{aligned} \tag{1.16}$$

with $Q_1 \in \mathrm{Hom}(H_{(0,0,0)}, H_{(0,0,0)}), Q_2 \in \mathrm{Hom}(H_{(0,0,2)}, H_{(0,0,2)})$. In the situation considered here, one has even $Q_1 = Q_2 = 0$, because the symbol L does not depend upon x. As a consequence of (1.15), (1.16), the multiplication of Op L from the left with Op S_1 reduces the singular perturbation (1.10) to a regular perturbation of the reduced problem:

$$\text{(1.17)} \qquad \mathrm{Op}\, S_1(\varepsilon,\xi)\ \mathrm{Op}\, L(\varepsilon,\xi) = \mathrm{Op}\, L(0,\xi) + \varepsilon Q_3$$

with $Q_3 \in \mathrm{Hom}(H_{(0,0,2)}, H_{(0,0,2)})$.
Since Op $L(0,\xi)$ is invertible, (1.17) implies that $(\mathrm{Op}\, L(\varepsilon,\xi))^{-1}$ can for $\varepsilon \in (0,\varepsilon_0]$ with ε_0 sufficiently small, be expanded in a convergent Neumann series which contains the known operators
Op S_1, Op $L(\varepsilon,\xi)$, $(\mathrm{Op}\, L(0,\xi))^{-1}$:

$$(\mathrm{Op}\, L(\varepsilon,\xi))^{-1} = \sum_{k\geq 0} ((\mathrm{Op}\, L(0,\xi))^{-1}(\mathrm{Op}\, L(0,\xi) - \mathrm{Op}\, S_1(\varepsilon,\xi)\mathrm{Op}\, L(\varepsilon,\xi)))^k \mathrm{Op}\, L(0,\xi)^{-1}$$
$$\mathrm{Op}\, S_1(\varepsilon,\xi).$$

Using finitely many terms in this series, high order asymptotic formulae for the solution of (1.10) can be obtained under very weak regularity assumptions upon f.

In the reduction of coercive singularly perturbed Wiener-Hopf equations to regular perturbations, the left reducing operator R_1^ε is a singularly perturbed Wiener-Hopf matrix operator. Its symbol can be obtained using the classical parametrix construction for the reduced problem.
If the spaces of the solutions to the perturbed and to the reduced problems are $\mathring{H}_{(s)}(U)$ and $\mathring{H}_{s_2}(U)$, respectively, the right reducing operator R_r^ε can be chosen as the pseudodifferential operator with the symbol $\varepsilon^{-s_1}(i\varepsilon\xi_N + \varepsilon|\xi'|+1)^{s_3}$, where ξ_N and ξ' denote the conormal and cotangential variables, respectively. If the spaces of the solutions are of the form $H_{(s)}(U)$, one can choose R_r^ε to be identity operator. As a consequence of

the coerciveness of the perturbed problem, the symbol of R_1^ε is invertible. Its inverse, which, by definition, is the symbol of S_1^ε, can be expressed in terms of the parametrix constructed in the proof of the two-sided a priori estimate.

A shortened version of this thesis will be published elsewhere ([W]).

The list of references given at the end is by far incomplete and mentions only the publications which, to the best of the author's knowledge, are directly connected with the topics of this paper. For a more complete bibliography, see [L], [T]. However, references to the publications in the field of asymptotic analysis which appeared during the last decade could, of course, not be included in [L], [T].

Following [Fr 1], the Sobolev type spaces $H_{(s)}$ with vectorial index $s \in \mathbb{R}^3$ will be defined and their basic properties will be stated without proof. This section is included in order to make the paper self-contained.

2.1 Definition of the spaces $H_{(s)}$

Let $s \in \mathbb{R}^3$ and let ε_0 be a positive constant.

Definition 2.1.1. ([Fr 1])

For a family of distributions $(0,\varepsilon_0] \times \mathbb{R}^{n-1} \ni (\varepsilon,\xi') \to u(\varepsilon,\xi',x_n) \in S'(\mathbb{R}_{x_n})$ let the norm $||u||_{(s),\xi'}$ be given by

$$(2.1.1) \qquad ||u||_{(s),\xi'} = ||\varepsilon^{-s_1}\langle\xi\rangle^{s_2}\langle\varepsilon\xi\rangle^{s_3}(F_{x_n\to\xi_n}u)(\varepsilon,\xi',\xi_n)||_{L^2(\mathbb{R}_{\xi_n})}$$

where $\xi = (\xi',\xi_n)$, $\langle\xi\rangle$ is given by

$$(2.1.2) \qquad \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^n,$$

and where

$$(2.1.3) \qquad F_{x_n\to\xi_n}u(\varepsilon,\xi',\xi_n) = \hat{u}(\varepsilon,\xi',\xi_n) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-i\xi_n x_n} u(\varepsilon,\xi',x_n)dx_n$$

is the Fourier transform of the function $u \in S(\mathbb{R}_{x_n})$ (and which is defined by duality if u is a generalized function).

We say that $u \in H_{(s),\xi'}(\mathbb{R})$ if

$$(2.1.4) \qquad \sup_{\varepsilon\in(0,\varepsilon_0]} ||u||_{(s),\xi'} < \infty.$$

Definition 2.1.2. [Fr 1]

For a family of distributions $(0,\varepsilon_0] \ni \varepsilon \to u(\varepsilon,x) \in S'(\mathbb{R}^n_x)$ let

$$(2.1.5) \qquad ||u||_{(s)} = ||\varepsilon^{-s_1} \langle\xi\rangle^{s_2} \langle\varepsilon\xi\rangle^{s_3} F_{x\to\xi} u(\varepsilon,\xi)||_{L^2(\mathbb{R}^n_\xi)}$$

where

$$(2.1.6) \qquad F_{x\to\xi} u(\varepsilon,\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi x} u(\varepsilon,x)dx$$

with $\xi x \overset{\text{def}}{=} \sum_{1\le i\le n} \xi_i x_i$ is the Fourier transform of the function $u \in S(\mathbb{R}^n)$, which is defined by duality if u is a generalized function.

We say that $u \in H_{(s)}(\mathbb{R}^n)$ if $\sup_{\varepsilon\in(0,\varepsilon_0]} ||u||_{(s)} < \infty$.

It was proved in [Fr 1] that $H_{(s)}(\mathbb{R}^n)$ is a Banach space.

Definition 2.1.3. [Fr 1]

For a family of functions $\psi : (0,\varepsilon_0] \times \mathbb{R}^{n-1} \to \mathbb{C}$ let

$$(2.1.7) \qquad [\psi]_{(s),\xi'} = \varepsilon^{-s_1} \langle\xi'\rangle^{s_2} \langle\varepsilon\xi'\rangle^{s_3} |\psi(\varepsilon,\xi')|$$

and let $\mathbb{C}_{(s),\xi'}$ be the space of functions ψ which satisfy the condition

$$\sup_{\varepsilon\in(0,\varepsilon_0]} [\psi]_{(s),\xi'} < \infty.$$

Let F^{-1} be the inverse Fourier transform:

$$F^{-1}_{\xi_n\to x_n} w = \overset{\vee}{w}(\varepsilon,\xi',x_n) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i\xi_n x_n} w(\varepsilon,\xi',\xi_n)d\xi_n$$

$$F^{-1}_{\xi\to x} w = \overset{\vee}{w}(\varepsilon,x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi x} w(\varepsilon,\xi)d\xi$$

and let the operator Π^+ be the Fourier transform of the multiplication operator with the characteristic function of the set

$$\mathbb{R}^n_+ = \{x = (x',x_n) \mid x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

For $w \in S(\mathbb{R})$, one has:

$$\Pi^+_{\xi_n} w(\xi',\xi_n) = \lim_{\delta \downarrow 0} (2\pi i)^{-1} \int_{\mathbb{R}} (\xi_n - \eta_n - i\delta)^{-1} w(\xi',\eta_n)d\eta_n .$$

It is well known (see, for instance, [V-E]), that Π^+ projects $L^2(\mathbb{R}^n_\xi)$ onto the subspace of those functions in $L^2(\mathbb{R}^n_\xi)$ which can be extended as analytic functions in the complex half plane $\mathrm{Im}\ \xi_n < 0$. For meromorphic functions $\xi_n \to w(\xi',\xi_n)$ which have no singularities on $\mathbb{R}$, one has:

$$\Pi^+_{\xi_n} w(\xi',\xi_n) = (2\pi i)^{-1} \int_{\gamma(\xi')} (\xi_n - \eta_n)^{-1} w(\xi',\eta_n)d\eta_n$$

where $\gamma(\xi')$ is a closed Jordan curve in the half plane $\{\mathrm{Im}\ t > 0\}$ which encloses all the singularities of the function $\eta_n \to w(\xi',\eta_n)$. Moreover, let $\Pi^- = \mathrm{Id} - \Pi_+$.

Definition 2.1.4. ([Fr 1])

Family of distributions

$$(0,\varepsilon_0] \times \mathbb{R}^{n-1} \ni (\varepsilon,\xi') \to u(\varepsilon,\xi',.) \in S'(\mathbb{R}_+)$$

is said to belong to the space $H_{(s),\xi'}(\mathbb{R}_+)$ if there exists an extension $lu(\varepsilon,\xi',.) \in S'(\mathbb{R})$ of u to $\mathbb{R}$ such that $\sup_{\varepsilon\in(0,\varepsilon_0]} ||lu||_{(s),\xi'} < \infty$. The norm in $H_{(s),\xi'}(\mathbb{R}_+)$ is given by

$$(2.1.8) \qquad ||u||^{(1)}_{(s),\xi'} = \inf_{l} \sup_{\varepsilon\in(0,\varepsilon_0]} ||lu||_{(s),\xi'}$$

where the infimum is taken over all extensions lu to $\mathbb{R}$. On $H_{(s),\xi'}$ we also introduce

$$||u||^{(2)}_{(s),\xi'} = \sup_{\varepsilon\in(0,\varepsilon_0]} ||u||^{+}_{(s),\xi'}$$

with

$$(2.1.9) \qquad ||u||^{+}_{(s),\xi'} = ||\varepsilon^{-s_1}\Pi^+_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}F_{x_n\to\xi_n}(lu)(\varepsilon,\xi',\xi_n)||_{L^2(\mathbb{R}_{\xi_n})}.$$

This norm does not depend upon the choice of the extension lu. Indeed, if l_1u, l_2u are two extensions of u to $\mathbb{R}$, then the support of l_1u-l_2u is contained in $\overline{\mathbb{R}}_-$. Therefore, the functions $F_{x_n\to\xi_n}(l_1u-l_2u)$ and $(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}F_{x_n\to\xi_n}(l_1u-l_2u)$ are analytic for $\mathrm{Im}\,\xi_n > 0$, so that

$$\Pi^+_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}F_{x_n\to\xi_n}(lu)(\varepsilon,\xi',\xi_n) \equiv 0.$$

Since $\Pi^+_{\xi_n}$ is an orthogonal projection with respect to the L^2-scalar product, one has

$$\begin{aligned}||u||^{(2)}_{(s),\xi'} &\le \inf_l ||\varepsilon^{-s_1}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3} \\ &\qquad F_{x_n\to\xi_n}(lu)(\varepsilon,\xi',\xi_n)||_{L^2(\mathbb{R}_{\xi_n})} \\ &\le ||u||^{(1)}_{(s),\xi'}.\end{aligned}$$

Thus, the mapping I given by $I(u) = u$ is bounded from $(H_{(s),\xi'}(\mathbb{R}_+), ||u||^{(1)}_{(s),\xi'})$ into $(H_{(s),\xi'}(\mathbb{R}_+), ||u||^{(2)}_{(s),\xi'})$. The closed graph theorem implies that the inverse operator I^{-1} is bounded, too. Thus, the norms $||u||^{(1)}_{(s),\xi'}$ and $||u||^{(2)}_{(s),\xi'}$ are equivalent.

<u>Definition 2.1.5.</u> ([Fr 1])

A family of distributions $(0,\varepsilon_0] \ni \varepsilon \to u(\varepsilon,x) \in S'(\mathbb{R}^n_x)$ is said to belong to the space $H_{(s)}(\mathbb{R}^n_+)$ if there exists an extension $lu \in S'(\mathbb{R}^n)$ of u to $\mathbb{R}^n$ such that $\sup_{\varepsilon\in(0,\varepsilon_0]} ||lu||_{(s)} < \infty$. The norm in $H_{(s)}(\mathbb{R}^n_+)$ is given by

$$\inf_l \sup_{\varepsilon\in(0,\varepsilon_0]} ||lu||_{(s)}$$

or, equivalently, by

$$\sup_{\varepsilon\in(0,\varepsilon_0]} ||u||^+_{(s)}$$

where

$$||u||^+_{(s)} = ||\varepsilon^{-s_1}\Pi^+_{\xi_n}(-i\xi_n+\langle\xi\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}F_{x\to\xi}(lu)(\varepsilon,\xi)||_{L^2(\mathbb{R}^n_\xi)}$$

does not depend upon the extension lu.

Let $U \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary ∂U. In order to define the spaces $H_{(s)}(U)$, let $(U_l)_{1\le l\le r}$ be a covering of ∂U by open (in $\mathbb{R}^n$) sets, such that there exist diffeomorphisms

$$\chi_l : U_l \cap U \to V_l, \quad 1 \le l \le r$$

where V_l is a subset of $\mathbb{R}^n_+$. Moreover, it is assumed that $\chi_l(\partial U \cap U_l) \subset \mathbb{R}^{n-1}\times\{0\}$. Let U_0 be an open subset of U such that $\overline{U}_0 \subset U$, $\bigcup_{0\le l\le r} U_l \supseteq U$, and let $(\psi_l)_{0\le l\le r}$ be a partition of unity subordinate to the covering $(U_l)_{0\le l\le r}$ of U. Moreover, let $l_0((u.\psi_l)\circ\chi_l^{-1}) : \mathbb{R}^n_+ \to \mathbb{C}$ and $l_0(u.\psi_0) : \mathbb{R}^n \to \mathbb{C}$ be the continuations by zero of the functions $(u.\psi_l)\circ\chi_l^{-1}$ and $u.\psi_0$ defined on the sets V_l and U_0, respectively.

Definition 2.1.6. ([Fr 1])

For a family of distributions $(0,\varepsilon_0] \to u(\varepsilon,.) \in S'(U)$ and for $s \in \mathbb{R}^3$, let

$$(2.1.10) \quad ||u||^2_{(s)} = \sum_{1\le l\le r} (||l_0(u.\psi_l)\circ\chi_l^{-1})||^+_{(s)})^2 + ||l_0(u.\psi_0)||^2_{(s)}$$

and let $H_{(s)}(U)$ be the space of distributions u which satisfy the condition $\sup_{\varepsilon\in(0,\varepsilon_0]} ||u||_{(s)} < \infty$.

Let $(U_l')_{0\le l\le r'}$ be a different covering of U with corresponding diffeomorphism χ_l' and a corresponding partition of unity $(\psi_l')_{0\le l\le r'}$. Let $||u||'_{(s)}$ be defined as in (2.1.10) with r,ψ_l,χ_l replaced with r',ψ_l',χ_l'. Then the norms $||u||_{(s)}$, $||u||'_{(s)}$ are equivalent uniformly with respect to ε:

$$C^{-1}||u||_{(s)} \le ||u||'_{(s)} \le C||u||_{(s)} \quad \forall\varepsilon \in (0,\varepsilon_0],\ \forall u \in H_{(s)}(U).$$

Let π_0 denote the restriction operator to the hyperplane $\mathbb{R}^{n-1}\times\{0\} \subset \mathbb{R}^n$.

Definition 2.1.7. ([Fr 1])

For a family of distributions $(0,\varepsilon_0] \ni \varepsilon \to \psi(\varepsilon,.) \in S'(\partial U)$ and for $s \in \mathbb{R}^3$, let

$$(2.1.11)\quad [\psi]^2_{(s)} = \sum_{1\le l\le r} ||\pi_0(l_0(\psi.\psi_l) \circ \chi_l^{-1})||^2_{(s)}$$

and let $H_{(s)}(\partial U)$ be the space of distributions ψ which satisfy the condition $\sup_{\varepsilon\in(0,\varepsilon_0]} [\psi]_{(s)} < \infty$.

Again, the norm in $H_{(s)}(\partial U)$ does not depend essentially upon the choice of the partition of unity.

Now we are going to define the spaces $\overset{\circ}{H}_{(s),\xi'}$, $\overset{\circ+}{H}_{(s)}$ with $s \in \mathbb{R}^3$ (see [V-E] where the spaces $\overset{\circ+}{H}_{s_2}$ with scalar index s_2 were introduced).

<u>Definition 2.1.8.</u>

A family of distributions $(0,\varepsilon_0] \times \mathbb{R}^{n-1} \ni (\varepsilon,\xi') \to u_+(\varepsilon,\xi',\circ) \in S'(\mathbb{R})$ is said to belong to the space $\overset{\circ+}{H}_{(s),\xi'}$ if $u \in H_{(s),\xi'}(\mathbb{R})$ and if supp $u_+(\varepsilon,\xi',\circ)$ is contained in $\overline{\mathbb{R}}_+$ for $\forall\varepsilon \in (0,\varepsilon_0]$.

<u>Definition 2.1.9.</u>

A family of distributions $(0,\varepsilon_0] \ni \varepsilon \to u_+(\varepsilon,\circ) \in S'(\mathbb{R}^n)$ is said to belong to the space $\overset{\circ+}{H}_{(s)}$ if $u_+ \in H_{(s)}(\mathbb{R}^n)$ and if supp $u_+(\varepsilon,\circ)$ is contained in $\overline{\mathbb{R}}^n_+$ for $\forall\varepsilon \in (0,\varepsilon_0]$.

<u>Definition 2.1.10.</u>

A family of distributions $(0,\varepsilon_0] \ni \varepsilon \to u(\varepsilon,\circ) \to S'(\mathbb{R}^n)$ is said to belong to the space $\overset{\circ}{H}_{(s)}(U)$ if $u \in H_{(s)}(\mathbb{R}^n)$ and if supp $u(\varepsilon,\circ)$ is contained in $\overline{U}$ for $\forall\varepsilon \in (0,\varepsilon_0]$.

2.2 Basic properties of the spaces $H_{(s)}$

Theorem 2.2.1. ([Fr 1])

The inequality

(2.2.1) $\quad ||u||_{(s')} \le C||u||_{(s)} \qquad \forall u \in H_{(s)}(\mathbb{R}^n)$

with a constant C independent upon ε,ξ',u, holds iff $s,s' \in \mathbb{R}^3$ satisfy the condition

(2.2.2) $\quad s_1 \ge s_1', \quad s_1+s_2 \ge s_1'+s_2', \quad s_2+s_3 \ge s_2'+s_3'.$

Proof. See the proof of Proposition 1.1.3 in [Fr 1]. □

We use the notation $s \succcurlyeq s'$ for writing (2.2.2) in a short way and the notation $s \succ s'$ if there is at least one strict inequality in (2.2.2). As an immediate consequence of Theorem 2.2.1, one has

Corollary 2.2.2. ([Fr 1])

For $s \succcurlyeq s'$, one has $H_{(s),\xi'}(\mathbb{R}) \subseteq H_{(s'),\xi'}(\mathbb{R})$, $H_{(s)}(\mathbb{R}^n) \subseteq H_{(s')}(\mathbb{R}^n)$ and the embedding operators are bounded uniformly with respect to $\xi' \in \mathbb{R}^{n-1}$.

An analogous embedding theorem holds for the spaces $H_{(s)}(U)$.

Theorem 2.2.3. ([Fr 1])

Let $\sigma,s \in \mathbb{R}^3$, $s \prec \sigma$ and

(2.2.3) $\quad \rho = s-t(\sigma-s), \qquad t \ge 0.$

Then for $\forall\delta > 0$ there exists a constant $C_\delta > 0$, such that

(2.2.4) $\quad ||u||^2_{(s)} \le \delta||u||^2_{(\sigma)} + C_\delta||u||^2_{(\rho)} \quad \forall u \in H_{(\sigma)}(\mathbb{R}^n).$

Proof. See the proof of Proposition 1.1.4 in [Fr 1]. □

An analogous interpolation theorem can be formulated and proved for the spaces $H_{(s)}(U)$.

Let the vectors $e_2, e \in \mathbb{R}^3$ be defined by

(2.2.5) $\quad e_2 = (0,1,0), \quad e = (1,-1,1).$

With π_0 the restriction operator to $\{x_n = 0\}$, the trace theorem for Sobolev spaces with vectorial order is stated as follows:

<u>Theorem 2.2.4</u>. ([Fr 1])

Let $s \in \mathbb{R}^3$ with $s_2+s_3 > \frac{1}{2}$. Then there exists a constant C independent upon $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $u \in H_{(s),\xi'}(\mathbb{R})$, such that:

$$[\pi_0 u]_{(s-\frac{1}{2}e_2),\xi'} \le C||u||_{(s),\xi'} \qquad \forall u \in H_{(s),\xi'}(\mathbb{R}), \text{ if } s_2 > \tfrac{1}{2}$$

$$[\pi_0 u]_{(s-\frac{1}{2}e_2),\xi'} \le C(1+|\ln \varepsilon|)||u||_{(s),\xi'} \qquad \forall u \in H_{(s),\xi'}(\mathbb{R}), \text{ if } s_2 = \tfrac{1}{2}$$

$$[\pi_0 u]_{(s-\frac{1}{2}e_2+(s_2-\frac{1}{2})e),\xi'} \le C||u||_{(s),\xi'} \qquad \forall u \in H_{(s),\xi'}(\mathbb{R}), \text{ if } s_2 < \tfrac{1}{2}.$$

<u>Proof</u>. See the proof of Lemmas 1.2.1, 1.2.2 and 1.2.3 in [Fr 1]. $\square$

An analogous trace theorem can be formulated and proved for the spaces $H_{(s)}(U)$.

<u>Lemma 2.2.5</u>.

For $u \in H_{(s),\xi'}(\mathbb{R}_+)$ with $s_2 \ge 0$, $s_3 \ge 0$ and for $\delta \in [\max(\frac{1}{2}-s_2,0),\frac{1}{2}]$, the following inequalities hold:

$$|\widehat{1_0 u}(\xi',\xi_n)| \le C\varepsilon^{s_1}\langle\xi'\rangle^{\frac{1}{2}-s_2-\delta}|\mathrm{Im}\,\xi_n|^{-1+\delta}\langle\varepsilon\xi'\rangle^{-s_3}||u||^+_{(s),\xi'}$$

$$\forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi' \in \mathbb{R}^{n-1},\ \forall\xi_n \in \mathbb{C},\ \mathrm{Im}\,\xi_n < 0,\ \forall u \in H_{(s),\xi'}(\mathbb{R}_+),$$

where C is a constant which does not depend upon ε,ξ, and u.

Proof. Since $\widehat{1_0 u}(\xi)$ is analytic for $\operatorname{Im}\xi_n < 0$, one has:

$$\widehat{1_0 u}(\xi',\xi_n) = (2\pi i)^{-1} \int_{\mathbb{R}} (\xi_n-\eta_n)^{-1} \widehat{1_0 u}(\xi',\eta_n)d\eta_n$$

$$= (2\pi i)^{-1} \langle\xi'\rangle^{\frac12-s_2-\delta} \langle\varepsilon\xi'\rangle^{-s_3} \int_{\mathbb{R}} ((\xi_n-\eta_n)^{-1}(-i\eta_n+\langle\xi'\rangle)^{\delta-\frac12}) \cdot$$

$$\cdot (-i\eta_n+\langle\xi'\rangle)^{\frac12-\delta}\langle\xi'\rangle^{s_2+\delta-\frac12} \langle\varepsilon\xi'\rangle^{s_3} \widehat{1_0 u}(\xi',\eta_n)d\eta_n.$$

The Cauchy-Schwarz inequality yields:

$$|\widehat{1_0 u}(\xi',\xi_n)| \le C\varepsilon^{s_1}\langle\xi'\rangle^{\frac12-s_2-\delta} \langle\varepsilon\xi'\rangle^{-s_3}(I(\varepsilon,\xi))^{\frac12} \cdot$$

$$\cdot \|\varepsilon^{-s_1}(-i\xi_n+\langle\xi'\rangle)^{\frac12-\delta}\langle\xi'\rangle^{s_2+\delta-\frac12} \langle\varepsilon\xi'\rangle^{s_3} \widehat{1_0 u}(\xi)\|_{L_2(\mathbb{R}_{\xi_n})}$$

where

$$I(\varepsilon,\xi) = \int_{\mathbb{R}} |\xi_n-\eta_n|^{-2}|-i\eta_n+\langle\xi'\rangle|^{2\delta-1}d\eta_n.$$

Using the substitution $\eta_n = t|\operatorname{Im}\xi_n|$, this integral is estimated as follows:

$$I(\varepsilon,\xi) \le \frac{C}{|\operatorname{Im}\xi_n|^{2-2\delta}} \int_{\mathbb{R}} \frac{dt}{(|\operatorname{Im}\xi_n|^{-1}\operatorname{Re}\xi_n-t|+1)^2\,|-it+|\operatorname{Im}\xi_n|^{-1}\langle\xi'\rangle|^{1-2\delta}}$$

$$\le C|\operatorname{Im}\xi_n|^{-2+2\delta}.$$

Moreover, one has

$$\|\varepsilon^{-s_1}(-i\xi_n+\langle\xi'\rangle)^{\frac12-\delta}\langle\xi'\rangle^{s_2-\frac12+\delta}\langle\varepsilon\xi'\rangle^{s_3} \widehat{1_0 u}(\xi)\|_{L^2(\mathbb{R}_{\xi_n})}$$

$$\le C\|\varepsilon^{-s_1}\langle\xi'\rangle^{s_2-\frac12+\delta} \langle\varepsilon\xi'\rangle^{s_3}\, 1_0 u(\xi',x_n)\|_{H_{\frac12-\delta,\xi'}(\mathbb{R}_{x_n})}$$

$$\le C\|\varepsilon^{-s_1}\langle\xi'\rangle^{s_2-\frac12+\delta} \langle\varepsilon\xi'\rangle^{s_3}\, u(\xi',x_n)\|^+_{(0,\frac12-\delta,0),\xi'}$$

$$\le C\|u\|^+_{(s),\xi'}$$

because the extension operator by zero is continuous from $H_{(0,s,0),\xi'}(\mathbb{R}_+)$ to $H_{(0,s,0),\xi'}(\mathbb{R})$ for $|s| < \frac12$ (see [V-E]).

This proves Lemma 2.2.5. □

As it is well known, the extension operator by zero is bounded from $H_{s_2}(\mathbb{R}_+)$ to $H_{s_2}(\mathbb{R})$ if $|s_2| < \frac{1}{2}$ (see [Ste] and the references there). We are going to extend this result to Sobolev type spaces with vectorial order.

Lemma 2.2.6.

Let $s \in \mathbb{R}^3$ be such that $|s_2| < \frac{1}{2}$, $|s_2+s_3| < \frac{1}{2}$. Then the extension operator by zero l_0 satisfies the following inequality:

$$(2.2.6) \qquad ||l_0 u||_{(s),\xi'} \le C||u||^+_{(s),\xi'} \quad \forall u \in H_{(s),\xi'}(\mathbb{R}_+)$$

with a constant C which does not depend upon u,ε,ξ'.

Proof: Let $lu \in H_{(s),\xi'}(\mathbb{R})$ be an extension of u which satisfies

$$(2.2.7) \qquad ||lu||_{(s),\xi'} \le C||u||^+_{(s),\xi'}$$

with C independent upon u,ε,ξ'. Then one has:

$$(2.2.8) \qquad ||l_0 u||_{(s),\xi'} \le C||\varepsilon^{-s_1}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\widehat{l_0 u}(\xi)||_{L^2(\mathbb{R}_{\xi_n})}$$

$$\le C||\varepsilon^{-s_1}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\,\Pi^+_{\xi_n}\,\widehat{lu}(\xi)||_{L^2(\mathbb{R}_{\xi_n})}$$

$$\le C\varepsilon^{-s_1}||T(lu)(\xi)+\Pi^+_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\,\widehat{lu}(\xi)||_{L^2(\mathbb{R}_{\xi_n})}$$

where

$$T(lu)(\xi) = (-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\,\Pi^+_{\xi_n}\,\widehat{lu}(\xi) -$$

$$- \Pi^+_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\,\widehat{lu}(\xi).$$

Since $\Pi^+_{\xi_n} = -H_{\xi_n} + \frac{1}{2}\mathrm{Id}$ with

$$H_{\xi_n} w(\xi) \overset{\text{def}}{=} (2\pi i)^{-1}\ \text{v.p.}\int_{\mathbb{R}} (\xi_n-\eta_n)^{-1}\, w(\xi',\eta_n)d\eta_n,$$

one has:

$$(Tw)(\xi) = H_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3} w(\xi) -$$
$$- (-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3} H_{\xi_n} w(\xi).$$

Following [Ste], we are now going to show that

$$\text{(2.2.9)} \qquad ||T(lu)(\xi)||_{L^2(\mathbb{R}_{\xi_n})} \le C||(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}\widehat{lu}(\xi)||_{L^2(\mathbb{R}_{\xi_n})}$$

holds with C independent upon u,ε,ξ'. One has

$$T(lu)(\xi) = \int_{\mathbb{R}} K(\varepsilon,\xi',\xi_n,\eta_n)\ w(\varepsilon,\xi',\eta_n)d\eta_n$$

where

$$w(\varepsilon,\xi',\eta_n) \overset{\text{def}}{=} (-i\eta_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\eta_n+\langle\varepsilon\xi'\rangle)^{s_3}\widehat{lu}(\xi',\eta_n)$$

and

$$K(\varepsilon,\xi',\xi_n,\eta_n) \overset{\text{def}}{=} \left(1-\frac{(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}}{(-i\eta_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\eta_n+\langle\varepsilon\xi'\rangle)^{s_3}}\right)(\xi_n-\eta_n)^{-1}.$$

Substituting $\eta_n = \xi_n\cdot\lambda$, one finds that

$$\int|T(lu)(\xi)|^2d\xi_n = \int\left|\int K(\varepsilon,\xi',\xi_n,\lambda\xi_n)\ w(\varepsilon,\xi',\lambda\xi_n)\xi_n d\lambda\right|^2 d\xi_n.$$

Further, with $\zeta_n = \lambda\xi_n$, one obtains:

$$\text{(2.2.10)} \quad \int|T(lu)(\xi)|^2d\xi_n = \int\left|\int K_1(\varepsilon,\xi',\zeta_n,\lambda)d\lambda\right|^2|w(\varepsilon,\xi',\zeta_n)|^2d\zeta_n$$

with

$$K_1(\varepsilon,\xi',\zeta_n,\lambda) \overset{\text{def}}{=} \left(1-\frac{(-i\lambda^{-1}\zeta_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\lambda^{-1}\zeta_n+\langle\varepsilon\xi'\rangle)^{s_3}}{(-i\zeta_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\zeta_n+\langle\varepsilon\xi'\rangle)^{s_3}}\right)(1-\lambda)^{-1}\lambda^{-\frac12}.$$

Since $s_2 < \frac12$, $s_2+s_3 < \frac12$, one has

$$\int_{-\frac12}^{\frac12}|K_1(\varepsilon,\xi',\zeta_n,\lambda)|d\lambda \le C < \infty$$

with C independent upon ε,ξ',ζ_n. Moreover, the conditions $s_2 > -\frac12$,

$s_2+s_3 > -\frac{1}{2}$ imply that, uniformly with respect to ε,ξ',ζ_n:

$$\left(\int_{-\infty}^{-\frac{1}{2}} + \int_{2}^{\infty}\right) |K_1(\varepsilon,\xi',\zeta_n,\lambda)|d\lambda \le C < \infty.$$

Finally, the uniform boundedness of $[\frac{1}{2},2] \ni \lambda \to |K_1(\varepsilon,\xi',\zeta_n,\lambda)|$ with respect to ε,ξ',ζ_n, implies that

$$\int_{\frac{1}{2}}^{2} |K_1(\varepsilon,\xi',\zeta_n,\lambda)|d\lambda \le C < \infty.$$

Hence,

$$\left|\int_{\mathbb{R}} K_1(\varepsilon,\xi',\zeta_n,\lambda)d\lambda\right| \le C < \infty$$

and (2.2.10) implies (2.2.9). As a consequence of (2.2.7) and (2.2.9), one obtains

$$\varepsilon^{-s_1}||T(lu)||_{L^2(\mathbb{R}_{\xi_n})} \le C||lu||_{(s),\xi'} \le C||u||^+_{(s),\xi'}.$$

(2.2.6) now follows from (2.2.8) and the last inequality. □

For $|s_2| < \frac{1}{2}$, $|s_2+s_3| < \frac{1}{2}$, Lemma 2.2.6 and the inequality

$$||u||^+_{(s),\xi'} = \inf_{l} ||lu||_{(s),\xi'} \le ||l_0u||_{(s),\xi'}$$

imply that $l_0 \in \mathrm{Iso}(H_{(s),\xi'}(\mathbb{R}_+),\overset{\circ}{H}{}^+_{(s),\xi'})$. Therefore, if s satisfies the above condition, the spaces $H_{(s),\xi'}(\mathbb{R}_+)$ and $\overset{\circ}{H}{}^+_{(s),\xi'}$ can be identiefied.

3.1 Definitions and auxiliary results

First, we recall several definitions and lemmas from [Fr 1], [Fr-W].

Definition 3.1.1. ([Fr 1])

A function $Q : (0,\varepsilon_0] \times \mathbb{R}^n \to \mathbb{C}$ is said to belong to the class P_ν for $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{R}^3$ if

(i) $Q(\varepsilon,\xi)$ is a polynomial in ξ, the coefficients of which depend continuously upon $\varepsilon \in (0,\varepsilon_0]$.

(ii) There exists a decomposition $Q = Q_0+R$, such that

$$(3.1.1) \qquad Q_0(\varepsilon,\xi) = \varepsilon^{-(\nu_1+\nu_2)} Q_0(1,\varepsilon\xi) \qquad \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n$$

$$(3.1.2) \qquad |Q_0(\varepsilon,\xi)| \le C\varepsilon^{-\nu_1} |\xi|^{\nu_2} \langle\varepsilon\xi\rangle^{\nu_3} \qquad \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n$$

$$(3.1.3) \qquad |R(\varepsilon,\xi)| \le C\varepsilon^{-\nu_1}(|\xi|^{-1}+\varepsilon)|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \quad \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n, |\xi| \ge 1$$

with some positive constants C,γ.

The function $Q_0(\varepsilon,\xi)$ will be called principal symbol of $Q(\varepsilon,\xi)$.

Since in section 3.2, symbols of the form $L(\varepsilon,\hat{\xi}',\xi_n)$ with $\hat{\xi}' = \langle\xi'\rangle|\xi'|^{-1}\xi'$ will be considered, we introduce here a class of symbols defined for $\xi \in \mathbb{R}^n | S$, where S is a set of measure zero.

Definition 3.1.2. ([Fr-W])

A function $L : (0,\varepsilon_0]\times\mathbb{R}^n\setminus S \to \mathbb{C}$, with S a set of measure zero, is said to belong to the class $L_\nu = L_\nu(\mathbb{R}^n)$ for $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{R}^3$ if

(i) there exist constants C,m such that

$$(3.1.4) \qquad |L(\varepsilon,\xi)| \le C\varepsilon^{-\nu_1}\langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \quad \forall(\varepsilon,\xi) \in (0,\varepsilon_0]\times\mathbb{R}^n\setminus S$$

$$(3.1.5) \qquad |L(\varepsilon,\xi)-L(\varepsilon,\eta)| \le C\langle\xi-\eta\rangle^m\, \varepsilon^{-\nu_1}\langle\xi\rangle^{\nu_2-1}\langle\varepsilon\xi\rangle^{\nu_3}$$

$$\forall(\varepsilon,\xi,\eta) \in (0,\varepsilon_0]\times\mathbb{R}^n\setminus S\times\mathbb{R}^n\setminus S$$

(ii) there exists a decomposition $L = L_0+R$ such that

$$(3.1.6) \qquad L_0(\varepsilon,\xi) = \varepsilon^{-(\nu_1+\nu_2)}L_0(1,\varepsilon\xi) \quad \forall(\varepsilon,\xi) \in (0,\varepsilon_0]\times\mathbb{R}^n\setminus S$$

$$\left.\begin{array}{ll}(3.1.7) & |L_0(\varepsilon,\xi)| \le C\varepsilon^{-\nu_1}|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \\ (3.1.8) & |R(\varepsilon,\xi)| \le C\varepsilon^{\nu_1}(|\xi|^{-1}+\varepsilon)|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}\end{array}\right\} \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n\setminus S,\ |\xi| \ge 1$$

with a positive constant C.

The function L_0 will be called the principal symbol of L. A function satisfying (3.1.6) is said to be homogeneous of degree $\nu_1+\nu_2$ in (ε^{-1},ξ).

<u>Lemma 3.1.3.</u>

(i) If $L_j \in L_{\nu^{(j)}}$ for $j = 1,2$, then $L_1\cdot L_2 \in L_{\nu^{(1)}+\nu^{(2)}}$.

(ii) If $L_1 \in L_{\nu^{(1)}}$ satisfies the inequality

$$(3.1.9) \qquad |L_1(\varepsilon,\xi)| \ge C\varepsilon^{-\nu_1^{(1)}}\langle\xi\rangle^{\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}} \quad \forall(\varepsilon,\xi) \in (0,\varepsilon_0]\times\mathbb{R}^n\setminus S_1$$

with a positive constant C, then $L_1(\varepsilon,\xi)^{-1} \in L_{-\nu^{(1)}}$.

<u>Proof</u>. For $j = 1,2$, let $S_j \subset \mathbb{R}^n$ be the sets where L_j is not defined, m_j the constants in the inequality (3.1.5) and let $L_j = L_{j0}+R_j$ be the decomposition which exists according to Definition 3.1.2 (ii). It is obvious that the symbol $L = L_1\cdot L_2$ satisfies (3.1.4) with $\nu = \nu^{(1)}+\nu^{(2)}$ on $\mathbb{R}^n\setminus S$ with $S = S_1 \cup S_2$.

In order to prove (3.1.5), one estimates $|L(\varepsilon,\xi)-L(\varepsilon,\eta)|$ as follows:

$$|L(\varepsilon,\xi)-L(\varepsilon,\eta)| = |L_1(\varepsilon,\xi)(L_2(\varepsilon,\xi)-L_2(\varepsilon,\eta)) + (L_1(\varepsilon,\xi)-L_1(\varepsilon,\eta))L_2(\varepsilon,\eta)|$$
$$\le C\varepsilon^{-\nu_1^{(1)}}\langle\xi\rangle^{\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}}\langle\xi-\eta\rangle^{m_2}\varepsilon^{-\nu_1^{(2)}}\langle\xi\rangle^{\nu_2^{(2)}-1}\langle\varepsilon\xi\rangle^{\nu_3^{(2)}} +$$
$$+ C\varepsilon^{-\nu_1^{(1)}}\langle\xi-\eta\rangle^{m_1}\langle\xi\rangle^{\nu_2^{(1)}-1}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}}\varepsilon^{-\nu_1^{(2)}}\langle\eta\rangle^{\nu_2^{(2)}}\langle\varepsilon\eta\rangle^{\nu_3^{(2)}}.$$

Since the inequality

$$\langle\xi\rangle^{\zeta}\langle\eta\rangle^{-\zeta} \le C_\zeta\langle\xi-\eta\rangle^{|\zeta|}, \qquad \xi,\eta \in \mathbb{R}^n,\ \zeta \in \mathbb{R}$$

implies that

$$\langle\eta\rangle^{\nu_2^{(2)}}\langle\varepsilon\eta\rangle^{\nu_3^{(2)}} \le C\langle\xi-\eta\rangle^{|\nu_2^{(2)}|+|\nu_3^{(2)}|}\langle\xi\rangle^{\nu_2^{(2)}}\langle\varepsilon\xi\rangle^{\nu_3^{(2)}},$$

one has (3.1.5) with $m = \max(m_2, m_1+|\nu_2^{(2)}|+|\nu_3^{(2)}|)$.

It is obvious that the function $L_0 = L_{10}\, L_{20}$ satisfies the conditions (3.1.6), (3.1.7). The function $R = L-L_0$ can be estimated as follows:

$$(3.1.10) \qquad |R(\varepsilon,\xi)| \le |R_1(\varepsilon,\xi)R_2(\varepsilon,\xi)|+|R_1(\varepsilon,\xi)L_{20}(\varepsilon,\xi)|+|R_2(\varepsilon,\xi)L_{10}(\varepsilon,\xi)|.$$

For $|\xi| \ge 1$ and $\varepsilon \le \varepsilon_0$ one has $(|\xi|^{-1}+\varepsilon)^2 \le C(|\xi|^{-1}+\varepsilon)$, such that

$$|R_1(\varepsilon,\xi)R_2(\varepsilon,\xi)| \le C\varepsilon^{-(\nu_1^{(1)}+\nu_1^{(2)})}(|\xi|^{-1}+\varepsilon)\ |\xi|^{\nu_2^{(1)}+\nu_2^{(2)}}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}+\nu_3^{(2)}}$$

Obviously, the two last terms on the right hand side of (3.1.10) are bounded by $C\varepsilon^{-\nu_1}(|\xi|^{-1}+\varepsilon)\ |\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}$, too. This proves the first part of Lemma 3.1.3.

If $L_1(\varepsilon,\xi)$ satisfies (3.1.9), then obviously (3.1.4) holds with $L(\varepsilon,\xi) = (L_1(\varepsilon,\xi))^{-1}$ and $\nu = -\nu^{(1)}$. The difference $L(\varepsilon,\xi)-L(\varepsilon,\eta)$ can be estimated as follows:

$$|L(\varepsilon,\xi)-L(\varepsilon,\eta)| = |L_1(\varepsilon,\xi)-L_1(\varepsilon,\eta)|\ |L_1(\varepsilon,\xi)^{-1}L_1(\varepsilon,\eta)^{-1}|$$

$$\leq C\varepsilon^{-\nu_1^{(1)}}\langle\xi-\eta\rangle^{m_1}\langle\xi\rangle^{\nu_2^{(1)}-1}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}}\varepsilon^{2\nu_1^{(1)}}\langle\xi\rangle^{-\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{-\nu_3^{(1)}}\langle\eta\rangle^{-\nu_2^{(1)}}\langle\varepsilon\eta\rangle^{-\nu_3^{(1)}}$$

$$\leq C\varepsilon^{-\nu_1}\langle\xi-\eta\rangle^{m}\langle\xi\rangle^{\nu_2-1}\langle\varepsilon\xi\rangle^{\nu_3}$$

with $m = m_1+|\nu_2^{(1)}|+|\nu_3^{(1)}|$. For $|\xi| \geq 1$, $\xi \notin S_1$, the function $L_{10}(\varepsilon,\xi)$ can be estimated from below as follows:

$$\begin{aligned}|\varepsilon^{\nu_1^{(1)}}|\xi|^{-\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{-\nu_3^{(1)}}L_{10}(\varepsilon,\xi)| &\geq C|\varepsilon^{\nu_1^{(1)}}\langle\xi\rangle^{-\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{-\nu_3^{(1)}}(L_1(\varepsilon,\xi)-R_1(\varepsilon,\xi))|\\ &\geq 2C_1-C|\varepsilon^{\nu_1^{(1)}}\langle\xi\rangle^{-\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{-\nu_3^{(1)}}R_1(\varepsilon,\xi)|\\ &\geq 2C_1-C_2(|\xi|^{-1}+\varepsilon)\\ &\geq C_1\end{aligned}$$

for $|\xi| \geq r_0 >> 1$, $\varepsilon \leq \varepsilon_0 << 1$ with positive constants C_1,C_2.
Since (3.1.6) implies that $L_{10}(\varepsilon,\xi) = t^{\nu_1+\nu_2}L_{10}(\varepsilon t,t^{-1}\xi)$ $\forall t > 0$, one has

$$|L_{10}(\varepsilon,\xi)| \geq C_1\varepsilon^{-\nu_1^{(1)}}\langle\xi\rangle^{\nu_2^{(1)}}\langle\varepsilon\xi\rangle^{\nu_3^{(1)}} \quad \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n-S_1,\ |\xi| \geq 1.$$

The function $L_0(\varepsilon,\xi) = L_{10}(\varepsilon,\xi)^{-1}$ thus satisfies (3.1.6), (3.1.7).
The rest $R(\varepsilon,\xi) = L(\varepsilon,\xi)-L_0(\varepsilon,\xi)$ can be estimated as follows:

$$\begin{aligned}|R(\varepsilon,\xi)| &= |L_1(\varepsilon,\xi)-L_{10}(\varepsilon,\xi)|\ |L_1(\varepsilon,\xi)^{-1}L_{10}(\varepsilon,\xi)^{-1}|\\ &\leq C\varepsilon^{-\nu_1}(|\xi|^{-1}+\varepsilon)|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \quad \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n\backslash S_1,\ |\xi| \geq 1.\end{aligned}$$

□

<u>Definition 3.1.4.</u> ([K-N])

For $L \in \mathcal{L}_\nu$ the operator

$$((\mathrm{Op}\ L(\varepsilon,\xi))u)(x) = L(\varepsilon,-i\frac{\partial}{\partial x})u(x) = F^{-1}_{\xi\to x}\ L(\varepsilon,\xi)F_{x\to\xi}u, \quad u \in S(\mathbb{R}^n),$$

is called pseudo-differential operator with the symbol L.

We introduce the operator $L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})$ by

$$(L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})u) = F^{-1}_{\xi_n\to x_n} L(\varepsilon,\xi',\xi_n)F_{x_n\to\xi_n}u.$$

Definition 3.1.5. ([Fr 1])

An operator from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ is said to have the vectorial order $\nu \in \mathbb{R}^3$, if for $\forall s \in \mathbb{R}^3$ it can be extended to a bounded operator in $\mathrm{Hom}(H_{(s)}(\mathbb{R}^n), H_{(s-\nu)}(\mathbb{R}^n))$.

Definition 3.1.6. ([Fr-W])

The function $L^0(\xi) \in L_{(0,\nu_2,0)}$ is called reduced symbol of $L(\varepsilon,\xi) \in L_{(\nu_1,\nu_2,\nu_3)}$ if one has:

$$\varepsilon^{\nu_1}\langle\xi\rangle^{-\nu_2}\langle\varepsilon\xi\rangle^{-\nu_3}L(\varepsilon,\xi)-\langle\xi\rangle^{-\nu_2}L^0(\xi) \in L_{(-1,1,-1)}. \tag{3.1.11}$$

(ii) L^0 being the reduced symbol of L, the operator $\mathrm{Op}\, L^0$ is called the reduced operator of $\mathrm{Op}\, L$.

Lemma 3.1.7.

(i) If the reduced symbol exists, it is unique.

(ii) If $L \in L_\nu$ is such that its reduced symbol L^0 exists, and if $\nu_3 \geq 0$, then one has:

$$\varepsilon^{\nu_1}L(\varepsilon,\xi)-L^0(\xi) \in L_{(-1,\nu_2+1,\nu_3-1)}.$$

Proof. (i) The inclusion (3.1.11) yields $L^0(\xi) = \lim_{\varepsilon\downarrow 0} \varepsilon^{\nu_1}L(\varepsilon,\xi)$, so that L^0 is well defined.

(ii) The symbol $\varepsilon^{\nu_1}L-L^0$ can be rewritten as follows:

$$\varepsilon^{\nu_1}L(\varepsilon,\xi)-L^0(\xi) = \langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}(\varepsilon^{\nu_1}\langle\xi\rangle^{-\nu_2}\langle\varepsilon\xi\rangle^{-\nu_3}L-\langle\xi\rangle^{-\nu_2}L^0)+L^0(\xi)(\langle\varepsilon\xi\rangle^{\nu_3}-1).$$

Since $\langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \in L_{(0,\nu_2,\nu_3)}$, (3.1.11) and lemma 3.1.3 yield that the first term on the right hand side of the last formula is in $L_{(-1,\nu_2+1,\nu_3-1)}$. Since $\langle\varepsilon\xi\rangle^{\nu_3}-1 \in L_{(-1,1,\nu_3-1)}$ for $\nu_3 \geq 0$, the symbol $L^0(\xi)(\langle\varepsilon\xi\rangle^{\nu_3}-1)$ is in $L_{(-1,\nu_2+1,\nu_3-1)}$, too. □

Definition 3.1.8. ([Fr 1])

The symbol $L \in \mathcal{L}_\nu$ is called elliptic of order ν if its principal symbol L_0 satisfies the condition

$$|L_0(\varepsilon,\xi)| \geq C\varepsilon^{-\nu_1}|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3} \quad \forall(\varepsilon,\xi) \in (0,\varepsilon_0] \times \mathbb{R}^n\setminus\{0\}$$

with a positive constant C. Let $\Omega_{n-1} \overset{\text{def}}{=} \{\omega' \in \mathbb{R}^{n-1} \mid |\omega'| = 1\}$.

Lemma 3.1.9. ([Fr 1])

Let $Q \in P_\nu$ be a principal symbol of order ν, $Q_{00}(\xi) = \lim_{\lambda\to\infty} \lambda^{-(\nu_2+\nu_3)} Q_0(1,\lambda\xi)$ the principal homogeneous part of $Q_0(1,\xi)$ of order $\nu_2+\nu_3$ and $\lambda(\varepsilon,\xi)$ zero of the equation

$$(3.1.12) \quad Q(\varepsilon,\xi',\lambda) = 0$$

Then there exists a rational number $q < 0$ and $C(\omega') \neq 0$ such that with $\omega' = |\xi'|^{-1}\xi'$:

$$|\xi'|^{-1}\lambda(\varepsilon,\xi') = \lambda_{00}(\omega')+(\varepsilon|\xi'|)^{q}(C(\omega')+o(1)), \quad \rho = \varepsilon|\xi'| \to \infty,$$

where $\lambda_{00}(\omega')$ is a zero of the equation

$$Q_{00}(\omega',\lambda) = 0$$

and $o(1) \to 0$ uniformly with respect to $\omega' \in \Omega_{n-1}$ as $\rho \to \infty$.

Proof. See the proof of Lemma 3.1.1 in [Fr 1]. $\square$

Lemma 3.1.10. ([Fr 1])

Let $Q \in P_\nu$ be an elliptic symbol of order ν, $Q^0(\xi) = \lim_{\varepsilon\downarrow 0} \varepsilon^{\nu_1} Q(\varepsilon,\xi)$ its reduced symbol, and let $\lambda(\varepsilon,\xi')$ be a zero of the equation (3.1.12).

Then there exist rational numbers $q_1 > 0$, $q_2 > -1$, and $C_j(\omega') \neq 0$, $j = 1,2$, such that either

$$|\xi'|^{-1}\lambda(\varepsilon,\xi') = |\xi'|^{-1}\lambda^0(\xi')+\rho^{q_1}(C_1(|\xi'|^{-1}\xi')+o(1)),$$

$$\rho = \varepsilon|\xi'| \to 0$$

with $\lambda^0(\xi')$ zero of the equation

$$Q^0(\xi',\lambda^0(\xi')) = 0$$

or

$$\varepsilon\lambda(\varepsilon,\xi') = \rho^{-1}\mu+\zeta^{q_2}(C_2(|\xi'|^{-1}\xi')+o(1)), \qquad \rho = \varepsilon|\xi'| \to 0$$

with μ a non-vanishing zero of the equation

(3.1.13) $\quad Q_0(1,0,\mu) = 0.$

Proof. See the proof of Lemma 3.1.2 in [Fr 1]. □

Definition 3.1.11. ([Fr 1])

An elliptic principal symbol $Q_0 \in P_\nu$ of order ν is called properly elliptic, if $Q_{00}(\xi) = \lim_{\lambda\to\infty} \lambda^{-(\nu_2+\nu_3)} Q_0(1,\lambda\xi)$ is a properly elliptic polynomial of order $\nu_2+\nu_3$.

Lemma 3.1.12. ([Fr 1])

Let $Q_0 \in P_\nu$ be a properly elliptic principal symbol of order ν. Then the reduced symbol $Q_0^0(\xi) = \lim_{\varepsilon\downarrow 0} \varepsilon^{\nu_1} Q_0(\varepsilon,\xi)$ is properly elliptic, too.

Proof. See the proof of Lemma 3.1.5 in [Fr 1]. □

If $Q_0(\varepsilon,\xi)$ is a properly elliptic principal symbol of order ν, then ν_2,ν_3 are even nonnegative integers:

$$\nu_j = 2r_j, \quad 2 \le j \le 3.$$

Let $\lambda \to Q_0^+(\varepsilon,\xi',\lambda)$ be the factor of the polynomial $\lambda \to Q_0(\varepsilon,\xi',\lambda)$ that corresponds to the zeroes of Q_0 contained in the upper half plane $\{\mathrm{Im}\,\lambda > 0\}$, when $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $\varepsilon > 0$. Besides, $\lambda \to Q_0^+$ being well defined up to a coefficient depending on ε,ξ', the latter is supposed to be chosen such that

$$(3.1.14)\quad \lim_{\lambda\to\infty} \lambda^{-(r_2+r_3)} Q_0^+(\varepsilon,\xi',\lambda) = \varepsilon^{r_3}.$$

Lemma 3.1.13. ([Fr 1])

Let $Q_0^{(l)}(\varepsilon,\xi)$, $l=1,2$, be properly elliptic of order $\nu^{(l)} = (\nu_1^{(l)}, 2r_2^{(l)}, 2r_3^{(l)})$ and $\lambda \to Q_0^{(l)+}(\varepsilon,\xi',\lambda)$ the factor of $\lambda \to Q_0^{(l)}$ described above. Then $L_0^+(\varepsilon,\xi',\lambda) = Q_0^{(1)+}(\varepsilon,\xi',\lambda)(Q_0^{(2)+}(\varepsilon,\xi',\lambda))^{-1}$ is homogeneous in $(\varepsilon^{-1},\xi',\lambda)$ of order r_2, where $r_j = r_j^{(1)} - r_j^{(2)}$, $2 \le j \le 3$, and satisfies the inequalities:

$$(3.1.15)\quad C^{-1}(|\xi'|+|\lambda|)^{r_2}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} \le$$
$$\le |L_0^+(\varepsilon,\xi',\lambda)| \le C(|\xi'|+|\lambda|)^{r_2}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3},$$

for any $\xi' \in \mathbb{R}^{n-1}$, any $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\,\lambda \le 0$; here the constant C does not depend upon $\varepsilon \in (0,\infty)$, $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathbb{C}$, $\mathrm{Im}\,\lambda \le 0$.

Besides, for $\varepsilon \in (0,\varepsilon_0]$ with ε_0 sufficiently small and $|\xi'| \le 2$, the function $\lambda \to L_0^+(\varepsilon,\xi',\lambda)$ can be split into the product

$$(3.1.16)\quad L_0^+(\varepsilon,\xi',\lambda) = L_{01}^+(\varepsilon,\xi',\lambda)\, L_{02}^+(\varepsilon,\xi',\lambda), \quad \varepsilon \in (0,\varepsilon_0],\ |\xi'| \le 2$$

where the zeroes and singularities of $\lambda \to L_{01}^+$ are contained in a compact domain in the half plane $\{\mathrm{Im}\,\lambda > 0\}$ uniformly with respect to $\varepsilon \in (0,\varepsilon_0]$, $|\xi'| \le 2$ and the zeroes and singularities of $\lambda \to L_{02}^+$ multiplied by ε also are contained in a compact domain in the half plane $\{\mathrm{Im}\,\lambda > 0\}$ uniformly with respect to $\varepsilon \in (0,\varepsilon_0]$, $|\xi'| \le 2$; the functions $(\varepsilon^{-1},\xi',\lambda) \to L_{01}^+$, $(\varepsilon^{-1},\xi',\lambda) \to L_{02}^+$ can be extended as homogeneous functions of $(\varepsilon^{-1},\xi',\lambda) \in (0,\infty) \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{C}$, of order r_2 and 0, respectively.
The leading coefficients of $\lambda \to L_{01}^+$, $\lambda \to L_{02}^+$ being 1 and ε^{r_3}, respectively, the following inequalities hold:

$$C^{-1}(|\xi'|+|\lambda|)^{r_2} \le |L^+_{01}(\varepsilon,\xi',\lambda)| \le C(|\xi'|+|\lambda|)^{r_2}$$

$$C^{-1}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} \le |L^+_{02}(\varepsilon,\xi',\lambda)| \le C(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3}$$

for any $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \le 0$.

Proof. See the proof of Lemma 3.1.7. in [Fr 1]. □

Lemma 3.1.14. ([Fr 1])

(i) Under the assumptions of Lemma 3.1.13, there exist integers $\sigma_2, \sigma_3 \ge 0$ and functions $\lambda_j(\varepsilon,\xi')$, $\mu_j(\varepsilon,\xi')$ satisfying the inequalities

$$(3.1.17)\begin{cases} |\lambda_j(\varepsilon,\xi')| \le C|\xi'| & \forall\varepsilon \in (0,\varepsilon_0],\ \ \forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\} \\ \operatorname{Im}\lambda_j(\varepsilon,\xi') \ge C^{-1}|\xi'| & 1 \le j \le |r_2|+2\sigma_2 \end{cases}$$

$$(3.1.18)\begin{cases} |\mu_j(\varepsilon,\xi')| \le C\varepsilon^{-1}\langle\varepsilon\xi\ \rangle & \forall\varepsilon \in (0,\varepsilon_0],\ \ \forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}, \\ \operatorname{Im}\mu_j(\varepsilon,\xi') \ge C^{-1}\varepsilon^{-1}\langle\varepsilon\xi'\rangle & 1 \le j \le |r_3|+2\sigma_3 \end{cases}$$

with a constant C, such that:

$$(3.1.19)\quad L^+_{01}(\varepsilon,\xi',\lambda) =$$

$$= \prod_{1\le j\le |r_2|}(\lambda-\lambda_j(\varepsilon,\xi'))^{\operatorname{sgn} r_2} \prod_{1\le l\le\sigma_2}(\lambda-\lambda_{|r_2|+l}(\varepsilon,\xi'))(\lambda-\lambda_{|r_2|+\sigma_2+l}(\varepsilon,\xi'))^{-1}.$$

$$(3.1.20)\quad L^+_{02}(\varepsilon,\xi',\lambda) =$$

$$= \prod_{1\le j\le |r_3|}(\varepsilon\lambda-\varepsilon\mu_j(\varepsilon,\xi'))^{\operatorname{sgn} r_3} \prod_{1\le l\le\sigma_3}(\varepsilon\lambda-\varepsilon\mu_{|r_3|+l}(\varepsilon,\xi'))$$

$$(\varepsilon\lambda-\varepsilon\mu_{|r_3|+\sigma_3+l}(\varepsilon,\xi'))^{-1}.$$

(ii) There exist a number $\rho_1 > 0$ and compact Jordan curves Γ_1,Γ_2 in the half plane $\{\operatorname{Im}\lambda > 0\}$, such that for $\forall\rho = \varepsilon\langle\xi'\rangle \in (0,\rho_1]$, $\forall\omega' \in \Omega_{n-1}$, the curves $\langle\xi'\rangle\Gamma_1, \varepsilon^{-1}\Gamma_2$ are disjoint and such that all the zeroes and singularities of the rational functions $\lambda \to L^+_{01}(\rho,\omega',\lambda)$ and $\lambda \to L^+_{02}(1,\rho\omega',\lambda)$ are contained in the interior of the curves Γ_1,Γ_2, respectively.

(iii) For any given $\rho_0 > 0$, there exists a compact Jordan curve $\Gamma = \Gamma(\rho_0)$ in the half plane $\{\mathrm{Im}\ \lambda > 0\}$, such that for $\forall\rho \geq \rho_0$, $\forall\omega' \in \Omega_{n-1}$, all the zeroes and singularities of the rational function $\lambda \to L^+_0(\rho,\omega',\lambda)$ are contained in the interior of Γ.

<u>Proof</u>. (i) Let σ_j, $j = 2,3$, be given by $2\sigma_j+|r_j| = r_j^{(1)}+r_j^{(2)}$. According to the result established in [Fr 1] and formulated here as Lemma 3.1.10, there exist precisely $r_2^{(1)}$ roots of the algebraic equation $Q_0^{(1)}(\rho,\omega',\lambda(\rho,\omega')) = 0$, which are bounded when $\rho \downarrow 0$, uniformly with respect to $\omega' \in \Omega_{n-1}$. Moreover, there exist precisely $r_3^{(1)}$ roots of the equations $Q_0^{(1)}(\rho,\omega',\mu(\rho,\omega')) = 0$, which grow like ρ^{-1} when $\rho \downarrow 0$, uniformly with respect to $\omega' \in \Omega_{n-1}$. This yields the representations (3.1.19), (3.1.20) for the functions L^+_{0j}, $j = 1,2$.

(ii) Again, we apply the result established in [Fr 1] and stated here as Lemma 3.1.10. For $1 \leq j \leq |r_2|+2\sigma_2$, there exists the limits $\lim_{\rho\downarrow 0} \lambda_j(\rho,\omega') = \lambda_j^0(\omega')$ uniformly with respect to $\omega' \subset \Omega_{n-1}$. Since the ellipticity condition implies that $\mathrm{Im}\ \lambda_j^0(\omega') \geq C_1 > 0$ for $\forall\omega' \in \Omega_{n-1}$, one can choose a curve Γ_1 in the upper half plane, such that $\lambda_j^0(\omega')$ are contained in the interior of Γ_1 for $\forall\omega' \in \Omega_{n-1}$, $\forall j$, $1 \leq j \leq |r_2|+2\sigma_2$. Then there exists $\rho_1' > 0$, such that $\lambda_j(\rho,\omega')$ are contained in the interior of Γ_1 for $\forall\rho \in (0,\rho_1']$, $\forall\omega' \in \Omega_{n-1}$.

For $1 \leq j \leq |r_3|+2\sigma_3$, there exist the limits $\lim_{\rho\downarrow 0} \rho\mu_j(\rho,\omega') = \mu_j$ uniformly with respect to $\omega' \in \Omega_{n-1}$. Let Γ_2 be a curve in the upper half plane which encloses the numbers μ_j, $1 \leq j \leq |r_3|+2\sigma_3$. Then there exists $\rho_2' > 0$, such that $\rho\mu_j(\rho,\omega')$ are contained in the interior of Γ_2 for $\forall\rho \in (0,\rho_2']$, $\forall\omega' \in \Omega_{n-1}$.

The curves $\langle\xi'\rangle\Gamma_1$ and $\varepsilon^{-1}\Gamma_2$ are disjoint if $\rho = \varepsilon\langle\xi'\rangle \in (0,\rho_3']$ with

$\rho_3' = (2 \max_{\lambda\in\Gamma_1} |\lambda|)^{-1} \min_{\lambda\in\Gamma_2} |\lambda|$. Therefore, one can choose $\rho_1 = \min(\rho_1',\rho_2',\rho_3')$.

(iii) As a consequence of the result established in [Fr 1] and formulated here as Lemma 3.1.9, there exist the limits $\lim_{\rho\to\infty} \lambda_j(\rho,\omega') = \lambda_{j00}(\omega')$ for any zero $\lambda_j(\rho,\omega')$ of one of the equations $Q_0^{(l)}(\rho,\omega',\lambda_j(\rho,\omega')) = 0$, $l = 1,2$, and one has $\mathrm{Im}\,\lambda_{j00}(\omega') \geq C_1 > 0$ uniformly with respect to $\omega' \in \Omega_{n-1}$. Since $\lambda_j(\rho,\omega')$ are continuous functions of ρ,ω', there exists a contour Γ_3 in the upper half plane which encloses all the zeroes and singularities of $\lambda \to L_0^+(\rho,\omega',\lambda)$ for $\rho \in [\rho_0,\infty)$, $\omega' \in \Omega_{n-1}$. □

The following definition is a natural generalization of the smoothness condition d) in [V-E] to the symbols with a small parameter:

Definition 3.1.15.

A symbol $L(\varepsilon,\xi) \in \mathcal{L}_\nu$ is said to be in the class $\mathcal{D}_\nu^k$ if for any $\mu \in \mathbb{R}^3$ with $\mu_2+\nu_2 \leq k$ the product $\varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}L(\varepsilon,\xi)$ can be decomposed as follows:

$$\varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}L(\varepsilon,\xi) = b(\varepsilon,\xi)+r(\varepsilon,\xi)$$

where the function $b \in \mathcal{L}_{\mu+\nu}$ is analytic in the half plane $\{\mathrm{Im}\,\xi_n > 0\}$ and the function

$$r(\varepsilon,\xi) = \Pi_{\xi_n}^+ \varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}L(\varepsilon,\xi)$$

satisfies the following inequalities:

$$(3.1.21)\quad |r(\varepsilon,\xi)| \leq C\varepsilon^{-(\mu_1+\nu_1)}\langle\xi'\rangle^{\mu_2+\nu_2}\langle\varepsilon\xi'\rangle^{\mu_3+\nu_3}(\langle\xi\rangle^{-1}\langle\xi'\rangle+\langle\varepsilon\xi\rangle^{-1}\langle\varepsilon\xi'\rangle)$$

$$\forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi = (\xi',\xi_n) \in \mathbb{R}^n$$

where C is a constant.

Example 3.1.16. The symbol $(i\varepsilon\xi_n+2\langle\varepsilon\xi'\rangle)^{-1}(i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)$ is in $\mathcal{D}^0_{(0,0,0)}$, whereas $(i\varepsilon\xi_n+\langle 2\varepsilon\xi'\rangle)^{-1}(i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)$ belongs to $\mathcal{D}^1_{(0,0,0)}$.

The following result, being the analogue of Theorem 1.4 in [V-E] for symbols with small parameter, will be proved, using a combination of the techniques from [V-E], [Fr 1].

Lemma 3.1.17.

Let $L(\varepsilon,\xi) \in \mathcal{D}^k_\nu$, $0 \le s_2 < k+\frac{1}{2}$, $s_2+s_3 \ge 0$.

Then one has:

$$||\pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})1_0 u||^+_{(s-\nu),\xi'} \le C||u||^+_{(s),\xi'}, \quad \forall u \in H_{(s),\xi'}(\mathbb{R}_+)$$

with a constant C which does not depend upon ε,ξ', and u.

Proof. Let $1 : H_{(s),\xi'}(\mathbb{R}_+) \to H_{(s),\xi'}(\mathbb{R})$ be an extension operator which is uniformly bounded with respect to ε,ξ'. Following [V-E], the norm of the function $\pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})1_0$ is estimated as follows:

$$||\pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})1_0 u||^+_{(s-\nu),\xi'} \le$$

$$\le ||\pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})1u||^+_{(s-\nu),\xi'} + ||\pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})(\pi_- 1u)||^+_{(s-\nu),\xi'}$$

$$\le C||1u||_{(s),\xi'} + ||\Pi^+_{\xi_n} w||_{L^2(\mathbb{R}_{\xi_n})},$$

where

$$w \overset{\text{def}}{=} \varepsilon^{-(s_1-\nu_1)}(-i\xi_n+\langle\xi'\rangle)^{s_2-\nu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3-\nu_3}L(\varepsilon,\xi)\widehat{\pi_- 1u}(\xi).$$

Consider first the case $\rho = \varepsilon\langle\xi'\rangle \le \rho_1$, where $\rho_1 > 0$ will be fixed later.

Let

$$\varepsilon^{-(s_1-\nu_1)}(-i\xi_n+\langle\xi'\rangle)^{k-\nu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3-\nu_3}L(\varepsilon,\xi) = b(\varepsilon,\xi)+r(\varepsilon,\xi)$$

be the decomposition according to Definition 3.1.15.

With $\gamma = \gamma(\varepsilon,\xi')$ a closed curve in the half plane $\{\mathrm{Im}\ \eta_n > 0\}$, which encloses all the singularities of the integrand, with positive imaginary parts, one has:

$$\Pi'_{\xi_n} w =$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{(-i\eta_n+\langle\xi'\rangle)^{s_2-k}}{\xi_n-\eta_n}(b(\varepsilon,\xi',\eta_n)+r(\varepsilon,\xi',\eta_n))\widehat{\pi\_lu}(\xi',\eta_n)d\eta_n$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{(-i\eta_n+\langle\xi'\rangle)^{s_2-k}}{\xi_n-\eta_n} r(\varepsilon,\xi',\eta_n)\widehat{\pi\_lu}(\xi',\eta_n)d\eta_n$$

since both b and $\widehat{\pi\_lu}$ are analytic for $\mathrm{Im}\ \eta_n > 0$.

We are now going to use a technique introduced in [Fr 1]. Let $\rho_1 > 0$ and let the curves $\Gamma_1,\Gamma_2 \subset \{\mathrm{Im}\ t > 0\}$ be as in Lemma 3.1.14(ii). Using the substitutions $\eta_n = \langle\xi'\rangle t$, $\eta_n = \varepsilon^{-1}t$ and the fact that the curves $\langle\xi'\rangle\Gamma_1$ and $\varepsilon^{-1}\Gamma_2$ are disjoint, one obtains:

$$\Pi^+_{\xi_n} w =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\langle\xi'\rangle(-i\langle\xi'\rangle t+\langle\xi'\rangle)^{s_2-k}}{\xi_n-\langle\xi'\rangle t} r(\varepsilon,\xi',\langle\xi'\rangle t)\widehat{\pi\_lu}(\xi',\langle\xi'\rangle t)dt +$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\varepsilon^{-1}(-i\varepsilon^{-1}t+\langle\xi'\rangle)^{s_2-k}}{\xi_n-\varepsilon^{-1}t} r(\varepsilon,\xi',\varepsilon^{-1}t)\widehat{\pi\_lu}(\xi',\varepsilon^{-1}t)dt.$$

The estimates

$$\left\|\frac{\varepsilon^{-1}}{\xi_n-\varepsilon^{-1}t}\right\|_{L^2(\mathbb{R}_{\xi_n})} \le C\varepsilon^{-\frac12},\ \left\|\frac{\langle\xi'\rangle}{\xi_n-\langle\xi'\rangle t}\right\|_{L^2(\mathbb{R}_{\xi_n})} \le C\langle\xi'\rangle^{\frac12}$$

hold uniformly with respect to $t \in \Gamma_1 \cup \Gamma_2$. Since Lemma 2.2.5 yields

$$|\widehat{\pi\_lu}(\xi',\xi_n)| \le C\varepsilon^{s_1}\langle\xi'\rangle^{\frac12-s_2-\delta}(\mathrm{Im}\ \xi_n)^{-1+\delta}\langle\varepsilon\xi'\rangle^{-s_3}||u||^+_{(s),\xi'}, \mathrm{Im}\ \xi_n > 0,$$

$||\pi^+_{\xi_n} w||$ can (with $\delta = \min(k+\frac12-s_2,\frac12)$) be estimated as follows:

$$||\Pi^+_{\xi_n} w||_{L^2(\mathbb{R}_{\xi_n})} \le C||u||^+_{(s),\xi'}.$$

Hence, the claim of Lemma 3.1.17 holds for $\varepsilon\langle\xi'\rangle \le \rho_1$.

Consider now the case $\varepsilon\langle\xi'\rangle \ge \rho_1$. Following again [Fr 1], let $\Gamma = \Gamma(\rho_1)$ be the curve constructed in the proof of Lemma 3.1.14(iii).

As a consequence of Lemma 2.2.5,the estimates

$$|\widehat{\pi_- lu}(\xi',\langle\xi'\rangle t)| \le C\varepsilon^{s_1}\langle\xi'\rangle^{-\frac{1}{2}-s_2}\langle\varepsilon\xi'\rangle^{-s_3}||u||^+_{(s),\xi'}$$

hold uniformly with respect to $t \in \Gamma$. Hence,

$$||\Pi^+_{\xi_n} w||_{L^2(\mathbb{R}_{\xi_n})} =$$

$$= ||\frac{1}{2\pi}\int_\Gamma \frac{\langle\xi'\rangle\varepsilon^{-(s_1-\nu_1)}}{\xi_n-\langle\xi'\rangle t}(-i\langle\xi'\rangle t+\langle\xi'\rangle)^{s_2-\nu_2}(-i\varepsilon\langle\xi'\rangle t+\langle\varepsilon\xi'\rangle)^{s_3-\nu_3} \cdot$$

$$\cdot\ L(\varepsilon,\xi',\langle\xi'\rangle t)\widehat{\pi_- lu}(\xi',\langle\xi'\rangle t)dt||_{L^2(\mathbb{R}_{\xi_n})}$$

$$\le C||u||^+_{(s),\xi'}. \qquad \square$$

The proof of Lemma 3.1.18 below is analogous to the proof of the corresponding result in [V-E] for symbols without small parameter.

Lemma 3.1.18.

If $a^{(1)} \in \mathcal{D}^k_{\nu^{(1)}}$ and $a^{(2)} \in \mathcal{D}^k_{\nu^{(2)}}$, then $a^{(1)} \cdot a^{(2)} \in \mathcal{D}^k_{\nu^{(1)}+\nu^{(2)}}$.

Proof. As a consequence of Lemma 3.1.3, one has: $a^{(1)}a^{(2)} \in \mathcal{L}_{\nu^{(1)}+\nu^{(2)}}$.
Let $\mu \in \mathbb{R}^3$ be given such that $\mu_2+\nu_2^{(1)}+\nu_2^{(2)} \leq k$. Then one has:

$$a^{(1)} = (-i\xi_n+\langle\xi'\rangle)^{-(\mu_2+\nu_2^{(2)})}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{-(\mu_3+\nu_3^{(2)})}(b^{(1)}+r^{(1)})$$
$$a^{(2)} = (-i\xi_n+\langle\xi'\rangle)^{-(\mu_2+\nu_2^{(1)})}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{-(\mu_3+\nu_3^{(1)})}(b^{(2)}+r^{(2)})$$

where $b^{(j)} \in \mathcal{L}_{(\nu_1^{(j)},\mu_2+\nu_2^{(1)}+\nu_2^{(2)},\mu_3+\nu_3^{(1)}+\nu_3^{(2)})}$, $j = 1,2$, are analytic for $\mathrm{Im}\,\xi_n > 0$ and where $r^{(j)}$, $j = 1,2$, satisfy the inequalities

$$|r^{(j)}(\varepsilon,\xi)| \leq$$
$$\leq C\varepsilon^{-\nu_1^{(j)}}\langle\xi'\rangle^{\mu_2+\nu_2^{(1)}+\nu_2^{(2)}}\langle\varepsilon\xi'\rangle^{\mu_3+\nu_3^{(1)}+\nu_3^{(2)}}(\langle\xi\rangle^{-1}\langle\xi'\rangle+\langle\varepsilon\xi\rangle^{-1}\langle\varepsilon\xi'\rangle).$$

One has:

$$\varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}\, a^{(1)}a^{(2)} = b^{(3)}+r^{(3)}$$

where

$$b^{(3)} = \varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{-(\mu_2+\nu_2^{(1)}+\nu_2^{(2)})}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{-(\mu_3+\nu_3^{(1)}+\nu_3^{(2)})}b^{(1)}b^{(2)}$$
$$\in \mathcal{L}_{(\mu+\nu^{(1)}+\nu^{(2)})}$$

and where the function

$$r^{(3)} = \varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{-(\mu_2+\nu_2^{(1)}+\nu_2^{(2)})}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{-(\mu_3+\nu_3^{(1)}+\nu_3^{(2)})}$$
$$(b^{(1)}r^{(2)}+b^{(2)}r^{(1)}+r^{(1)}r^{(2)})$$

satisfies the inequality:

$$|r^{(3)}(\varepsilon,\xi)| \le \varepsilon^{-(\mu_1+\nu_1^{(1)}+\nu_1^{(2)})} <\xi'>^{\mu_2+\nu_2^{(1)}+\nu_2^{(2)}} <\varepsilon\xi'>^{\mu_3+\nu_3^{(1)}+\nu_3^{(2)}}$$

$$(<\xi>^{-1} <\xi'>+<\varepsilon\xi>^{-1}<\varepsilon\xi'>). \quad \square$$

For $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$, let $\omega' = |\xi'|^{-1}\xi'$ and $\hat{\xi}' = <\xi'>\omega'$.

Lemma 3.1.19.

Under the assumptions of Lemma 3.1.13, the symbols $a(\varepsilon,\xi) = (L_0^+(\varepsilon,\hat{\xi}',\xi_n))^{-1}L_0(\varepsilon,\hat{\xi}',\xi_n)$ and $b(\varepsilon,\xi) = L_{01}^+(\varepsilon,\hat{\xi}',\xi_n)$ belong to the classes $\mathcal{D}^\infty_{(\nu_1,\nu_2-r_2,\nu_3-r_3)}$ and $\mathcal{D}^\infty_{(0,r_2,0)}$, respectively. Moreover, if $r_3 \ge 0$ and $\sigma_3 = 0$ in (3.1.20), then $L_{02}^+ \in \mathcal{D}^\infty_{(0,0,r_3)}$.

Proof. Since the construction of L_0^+ implies that $a(\varepsilon,\xi',\xi_n)$ is analytic for $\operatorname{Im}\xi_n > 0$, the condition in Definition 3.1.15 is satisfied with $r(\varepsilon,\xi) \equiv 0$. Now it will be proved that for λ_j satisfying (3.1.17), one has $(\xi_n-\lambda_j(\varepsilon,\hat{\xi}'))^{-1} \in \mathcal{D}^\infty_{(0,-1,0)}$. With $\mu \in \mathbb{R}^3$ arbitrary and $c(\varepsilon,\xi) = \varepsilon^{-\mu_1}(-i\xi_n+<\xi'>)^{\mu_2}(-i\varepsilon\xi_n+<\varepsilon\xi'>)^{\mu_3}$, one has $c(\varepsilon,\xi)(\xi_n-\lambda_j(\varepsilon,\hat{\xi}'))^{-1} =$ $= b_1(\varepsilon,\xi)+r_1(\varepsilon,\xi)$ where $b_1(\varepsilon,\xi) = (c(\varepsilon,\xi)-c(\varepsilon,\xi',\lambda_j(\varepsilon,\hat{\xi}')))(\xi_n-\lambda_j(\varepsilon,\hat{\xi}'))^{-1}$ is analytic for $\operatorname{Im}\xi_n > 0$ and where $r_1(\varepsilon,\xi) = c(\varepsilon,\xi',\lambda_j(\varepsilon,\hat{\xi}'))(\xi_n-\lambda_j(\varepsilon,\hat{\xi}'))^{-1}$ satisfies (3.1.21) with $\nu = (0,-1,0)$. Since L_{01}^+ can be written in the form (3.1.19), Lemma 3.1.18 yields $b \in \mathcal{D}^\infty_{(0,r_2,0)}$. Finally, if $r_3 \ge 0$ and $\sigma_3 = 0$ in (3.1.20), then for $\forall\mu \in \mathbb{R}^3$, one has $r \equiv 0$ in Definition 3.1.15. $\square$

Lemma 3.1.20.

A symbol $L_{02}^+(\varepsilon,\xi)$ of the form (3.1.20), with μ_j satisfying (3.1.18) and $r_3 \ge 0$, is in the class $\mathcal{D}^k_{(0,0,r_3)}$ iff there exists a decomposition $L_{02}^+(1,\eta) = b(\eta)+r(\eta)$ where $b \in \mathcal{L}_{(0,0,r_3)}$ is a polynomial in η_n and where the function $r(\eta)$ satisfies the following inequality:

$$(3.1.22) \qquad |r(\eta)| \le C|\eta'|^k<\eta'>^{r_3+1-k}<\eta>^{-1} \quad \forall\eta = (\eta',\eta_n) \in \mathbb{R}^n$$

with a constant C which does not depend upon η.

Proof. Assume first that $L_{02}^{+}(\varepsilon,\xi) = L_{02}^{+}(1,\varepsilon\xi) = b(\varepsilon\xi)+r(\varepsilon\xi)$, where $b(\varepsilon\xi)$ is analytic for $\operatorname{Im}\xi_n > 0$ and where $r(\varepsilon\xi)$ satisfies (3.1.22). One has:

$$\varepsilon^{-\mu_1}(-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}L_{02}^{+}(\varepsilon,\xi) = b_1(\varepsilon,\xi)+r_1(\varepsilon,\xi) \tag{3.1.23}$$

where

$$\varepsilon^{\mu_1}b_1(\varepsilon,\xi) = (-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}b(\varepsilon,\xi) +$$

$$+\Pi_{\xi_n}^{-}((-i\xi_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{\mu_3}r(\varepsilon,\xi))$$

and

$$\varepsilon^{\mu_1}r_1(\varepsilon,\xi) = (2\pi i)^{-1}\int_{\gamma}(\xi_n-\eta_n)^{-1}(-i\eta_n+\langle\xi'\rangle)^{\mu_2}(-i\varepsilon\eta_n+\langle\varepsilon\xi'\rangle)^{\mu_3}r(\varepsilon\xi',\varepsilon\eta_n)d\eta_n$$

with $\gamma = \gamma(\varepsilon,\xi')$ a closed Jordan contour in the upper half plane which encloses all the singularities of $\eta_n \to r(\varepsilon,\xi',\eta_n)$. Consider first the case $\varepsilon|\xi'| \le 1$. Let Γ_1 be a curve in $\{\operatorname{Im} t > 0\}$ which encloses all the singularities of $t \to r(\eta',t)$ for $\forall|\eta'| \le 1$. Then one has:

$$\varepsilon^{\mu_1}r_1(\varepsilon,\xi) = (2\pi i)^{-1}\int_{\Gamma_1}(\varepsilon\xi_n-t)^{-1}(-i\varepsilon^{-1}t+\langle\xi'\rangle)^{\mu_2}(-it+\langle\varepsilon\xi'\rangle)^{\mu_3}r(\varepsilon\xi',t)dt$$

According to (3.1.20), one has $L_{02}^{+}(\varepsilon,\xi) = Q_1(\varepsilon,\xi)(Q_2(\varepsilon,\xi))^{-1}$ where Q_1, Q_2 are elliptic polynomials in ξ_n of orders $(0,0,r_3+\sigma_3)$ and $(0,0,\sigma_3)$, with $\sigma_3 \ge 0$, respectively. The polynomial Q_3 being defined by

$$Q_1(\varepsilon,\xi) \equiv Q_3(\varepsilon,\xi) \pmod{Q_2(\varepsilon,\xi)},\ \deg Q_3 \le \sigma_3-1,$$

one has:

$$r(\varepsilon,\xi) = \Pi_{\xi_n}^{+}\ L_{02}^{+}(\varepsilon,\xi) = Q_3(\varepsilon,\xi)(Q_2(\varepsilon,\xi))^{-1}.$$

The estimate (3.1.22) implies that:

$$|Q_3(\varepsilon,\xi)| \le C|\varepsilon\xi'|^{k}\langle\varepsilon\xi\rangle^{\sigma_3-1} \quad \text{for } |\varepsilon\xi'| \le 1,\ \xi \in \mathbb{R}^n$$
$$|Q_3(\varepsilon,\xi)| \le C\langle\varepsilon\xi'\rangle^{r_3+1}\langle\varepsilon\xi\rangle^{\sigma_3-1} \quad \text{for } |\varepsilon\xi'| \ge 1,\ \xi \in \mathbb{R}^n.$$

Since Q_3 is a polynomial in ξ_n, one has

$$|Q_3(\varepsilon,\xi)| \le C|\varepsilon\xi'|^k <\varepsilon\xi>^{\sigma_3-1} \qquad \text{for } |\varepsilon\xi'| \le 1,\ \xi' \in \mathbb{R}^{n-1},\ \xi_n \in \mathbb{C}$$
$$|Q_3(\varepsilon,\xi)| \le C<\varepsilon\xi'>^{r_3+1} <\varepsilon\xi>^{\sigma_3-1} \qquad \text{for } |\varepsilon\xi'| \ge 1,\ \xi' \in \mathbb{R}^{n-1},\ \xi_n \in \mathbb{C}.$$

Since $r(\varepsilon,\xi) = Q_3(\varepsilon,\xi)(Q_2(\varepsilon,\xi))^{-1}$ with Q_2 elliptic, (3.1.22) holds for $\eta_n \in \Gamma_1$.

Using the fact that Γ_1 is independent upon ε,ξ' for $\varepsilon|\xi'| \le 1$, one obtains for $\mu_2 \le k$:

$$|r_1(\varepsilon,\xi)| \le C\varepsilon^{-\mu_1}|\xi'|^{\mu_2}(\varepsilon|\xi'|)^{k-\mu_2}<\varepsilon\xi>^{-1} \quad \forall\varepsilon \in (0,\varepsilon_0],\ \xi' \in \mathbb{R}^{n-1}\setminus\{0\},$$
$$\varepsilon|\xi'| \le 1,\ \xi_n \in \mathbb{R}$$
$$\le C\varepsilon^{-\mu_1}|\xi'|^{\mu_2}<\varepsilon\xi'>^{\mu_3+r_3+1}<\varepsilon\xi>^{-1}$$

since $<\varepsilon\xi'> \sim 1$. Consider now the case $\varepsilon|\xi'| > 1$.

Let $\Gamma_2 \subset \{\mathrm{Im}\ t > 0\}$ be a closed curve which encloses all the singularities of $t \to r(\eta',|\eta'|t)$ for $|\eta'| > 1$. Then one has:

$$\varepsilon^{\mu_1} r_1(\varepsilon,\xi) = (2\pi i)^{-1} \int_{\Gamma_2} \varepsilon|\xi'|(\varepsilon\xi_n-\varepsilon|\xi'|t)^{-1}(-i|\xi'|t+<\xi'>)^{\mu_2}$$
$$(-i\varepsilon|\xi'|t+<\varepsilon\xi'>)^{\mu_3} r(\varepsilon\xi',\varepsilon|\xi'|t)dt.$$

Since for $\varepsilon|\xi'| \ge 1$, there is the equivalency

$$\varepsilon|\xi'| \le <\varepsilon\xi'> \le 2\varepsilon|\xi'|,$$

one obtains, using (3.1.22):

$$|r_1(\varepsilon,\xi)| \le C\varepsilon^{-\mu_1}<\xi'>^{\mu_2}<\varepsilon\xi'>^{\mu_3+r_3+1}<\varepsilon\xi>^{-1} \quad \forall\varepsilon \in (0,\varepsilon_0],$$
$$\forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\},\ \varepsilon|\xi'| \ge 1,\ \xi_n \in \mathbb{R}.$$

Therefore, $L^+_{02} \in \mathcal{D}^k_{(0,0,r_3)}$.

Assume not that $L^+_{02} \in \mathcal{D}^k_{(0,0,r_3)}$. Then one has (3.1.23) for $\mu_2 = k$, where $r_1(\varepsilon,\xi)$ satisfies (3.1.21) with $\nu = (0,0,r_3)$. Let $\gamma(\varepsilon,\xi')$ a curve in the half plane $\{\mathrm{Im}\ \xi_n > 0\}$ which encloses all the singularities of $\xi_n \to L^+_{02}(\varepsilon,\xi',\xi_n)$. The function r in the decomposition $L^+_{02} = b+r$ can be written as follows:

$$r(\varepsilon,\xi) = (2\pi i)^{-1}\int_\gamma (\xi_n-\eta_n)^{-1} L^+_{02}(\varepsilon,\xi',\eta_n)d\eta_n$$
$$= (2\pi i)^{-1}\int_\gamma (\xi_n-\eta_n)^{-1}\varepsilon^{\mu_1}(-i\eta_n+\langle\xi'\rangle)^{-k}(-i\varepsilon\eta_n+\langle\varepsilon\xi'\rangle)^{-\mu_3}(b_1+r_1)(\varepsilon,\xi',\eta_n)d\eta_n.$$

Since $\eta_n \to b_1(\varepsilon,\xi',\eta_n)$ is analytic inside the curve γ, one obtains after the substitution $\eta_n = \varepsilon^{-1}t$:

$$r(\varepsilon,\xi) = (2\pi i)^{-1}\int_{\Gamma_1} (\varepsilon\xi_n-t)^{-1}\varepsilon^{\mu_1+k}(-it+\varepsilon\langle\xi'\rangle)^{k}(-it+\langle\varepsilon\xi'\rangle)^{-\mu_3} r_1(\varepsilon,\xi',\varepsilon^{-1}t)dt$$

where Γ_1 is a curve in the half plane $\{\mathrm{Im}\ t > 0\}$ which does not depend upon $\varepsilon,\xi',\varepsilon|\xi'| \le 1$. Therefore, $|r(\varepsilon,\xi)|$ can for $\varepsilon|\xi'| \le 1$ be estimated as follows:

$$|r(\varepsilon,\xi)| \le C(\varepsilon|\xi'|)^k\langle\varepsilon\xi\rangle^{-1}$$

with a positive constant C.

For $\varepsilon|\xi'| \ge 1$, there exists a curve $\Gamma_2 \subset \{\mathrm{Im}\ t > 0\}$ which does not depend upon ε,ξ' and which encloses all the singularities of $t \to r_1(\varepsilon,\xi',|\xi'|t)$. One has:

$$r(\varepsilon,\xi) = (2\pi i)^{-1}\int_{\Gamma_2} \varepsilon|\xi'|(\varepsilon\xi_n-\varepsilon|\xi'|t)^{-1}(-i|\xi'|t+\langle\xi'\rangle)^{-k}$$
$$(-i\varepsilon|\xi'|t+\langle\varepsilon\xi'\rangle)^{-\mu_3} r_1(\varepsilon,\xi',|\xi'|t)dt$$

such that

$$|r(\varepsilon,\xi)| \le C\langle\varepsilon\xi'\rangle^{r_3+\mu_3+1}\langle\varepsilon\xi\rangle^{-1}$$

for $\varepsilon|\xi'| \ge 1$ with a positive constant C. □

<u>Lemma 3.1.21.</u>

Every symbol $L^+_{02} \in \mathcal{D}^k_{(0,0,r_3)}$ of the form (3.1.20), can be written as a product $L_{02}^{+}(\varepsilon,\xi) = c(\varepsilon,\xi).d(\varepsilon,\xi)$ where c is an elliptic polynomial in ξ_n of order $(0,0,r_3)$ with zeroes in the half plane $\{\mathrm{Im}\ \xi_n > 0\}$ and where d belongs to the class $\mathcal{D}^k_{(0,0,0)}$.

<u>Proof</u>. Consider first the case $k > 0$. According to Lemma 3.1.20, $L_{02}^{+}(\varepsilon,\xi)$ can be rewritten as follows:

$$L_{02}^{+}(\varepsilon,\xi) = L_{02}^{+}(1,\varepsilon\xi) = b(\varepsilon\xi)+r(\varepsilon\xi)$$

where b is a polynomial of order $(0,0,r_3)$ and where r satisfies (3.1.22). The zeroes of $t \to b(0,t) = L_{02}^{+}(1,0,t)$ are contained in the upper half plane $\{\mathrm{Im}\ t > 0\}$ because $L_{02}^{+}(\varepsilon,\xi)$ has the form (3.1.20). Since the zeroes of polynomials depend continuously upon the coefficients, for $\varepsilon|\xi'| < \rho_0 << 1$ the zeroes of $t \to b(\varepsilon\xi',t)$ are contained in $\{\mathrm{Im}\ t > 0\}$, as well. For $\varepsilon|\xi'| < \rho_0$, define $c(1,\varepsilon\xi',t) = b(\varepsilon\xi',t)$, $t \in \mathbb{R}$ and for $\varepsilon|\xi'| \geq \rho_0$ let $c(1,\varepsilon\xi)$ be an elliptic polynomial in ξ_n of order $(0,0,r_3)$ with zeroes in $\{\mathrm{Im}\ \xi_n > 0\}$ which for $\varepsilon|\xi'| = \rho_0$ coincides with $b(\varepsilon\xi)$. Finally let $c(\varepsilon,\xi) = c(1,\varepsilon\xi)$. Since $c^{-1}L_{02}^{+} = 1+c^{-1}r$ for $\varepsilon|\xi'| \leq \rho_0$ and since $c^{-1}r$ satisfies (3.1.22), one has $c^{-1}L_{02}^{+} \in \mathcal{D}^k_{(0,0,0)}$.
Consider now the case $k = 0$. One can choose $c(\varepsilon,\xi) = (\varepsilon\xi_n - i\langle\varepsilon\xi'\rangle)^{r_3}$. Indeed, for functions $\lambda_j(\varepsilon,\xi'),\lambda_k(\varepsilon,\xi')$ satisfying (3.1.18), one has:

$$(\varepsilon\xi_n-\varepsilon\lambda_j(\varepsilon,\xi'))(\varepsilon\xi_n-\varepsilon\lambda_k(\varepsilon,\xi'))^{-1} = 1+r(\varepsilon,\xi)$$

where

$$r(\varepsilon,\xi) = \varepsilon(\lambda_k(\varepsilon,\xi')-\lambda_j(\varepsilon,\xi'))(\varepsilon\xi_n-\varepsilon\lambda_k(\varepsilon,\xi'))^{-1}$$

satisfies (3.1.22) with $k = 0$. Therefore, $(\varepsilon\xi_n-\varepsilon\lambda_j(\varepsilon,\xi'))(\varepsilon\xi_n-\varepsilon\lambda_k(\varepsilon,\xi'))^{-1} \in \mathcal{D}^0_{(0,0,0)}$. Since $c^{-1}L_{02}^{+}$ can be written as a finite product of such symbols, Lemma 3.1.18 yields $c^{-1}L_{02}^{+} \in \mathcal{D}^0_{(0,0,0)}$. □

Lemma 3.1.22.

If the symbol $L_{02}^{+} \in \mathcal{D}^{k}_{(0,0,0)}$ has the form (3.1.20), the symbol $(L_{02}^{+})^{-1}$ belongs to the class $\mathcal{D}^{k}_{(0,0,0)}$, too.

Proof. As a consequence of Lemma 3.1.20, one has $L_{02}^{+}(\varepsilon,\xi) = 1+r(\varepsilon,\xi)$, where $r(\varepsilon,\xi)$ satisfies (3.1.22). Hence $L_{02}^{+}(\varepsilon,\xi)^{-1} = 1+r_1(\varepsilon,\xi)$ where $r_1 = -(L_{02}^{+})^{-1}r$ satisfies (3.1.22), too. Applying again Lemma 3.1.20, one concludes that $(L_{02}^{+})^{-1} \in \mathcal{D}^{k}_{(0,0,0)}$. □

Lemma 3.1.23.

If the symbol $L_{02} \in \mathcal{D}^{k}_{(0,0,r_3)}$ has the form (3.1.20) with $k > 0$, then $r_3 \geq 0$.

Proof. Assume that $r_3 < 0$. If $p(\varepsilon,\xi',\lambda)^{-1}$ is defined to be the first factor in the factorization (3.1.20), p is a polynomial in λ of order $(0,0,-r_3)$ and hence $p \in \mathcal{D}^{\infty}_{(0,0,-r_3)}$. Thus, Lemma 3.1.18 implies that $pL_{02} \in \mathcal{D}^{k}_{(0,0,0)}$ and Lemma 3.1.22 yields $p^{-1}L_{02}^{-1} \in \mathcal{D}^{k}_{(0,0,0)}$.
Applying Lemma 3.1.18 again, one gets $p^{-1} = p^{-1}L_{02}^{-1}L_{02} \in \mathcal{D}^{k}_{(0,0,r_3)}$. Since $p_1 = \prod_{2\leq j\leq |r_3|} (\varepsilon\lambda-\varepsilon\mu_j(\varepsilon,\xi')) \in \mathcal{D}^{\infty}_{(0,0,-r_3-1)}$, one has $(\varepsilon\xi_n-\varepsilon\mu_1(\varepsilon,\xi'))^{-1} = p^{-1}.p_1 \in \mathcal{D}^{k}_{(0,0,-1)}$ which is not true for $k > 0$.

Definition 3.1.24.

A rational function $\lambda \to k(\varepsilon,\xi',\lambda)$, whose singularities are located in the half plane $\{\mathrm{Im}\,\lambda > 0\}$, is called a Poisson symbol of order $\alpha \in \mathbb{R}^3$ if

$$(3.1.24)\qquad |k(\varepsilon,\xi)| \leq C\varepsilon^{-\alpha_1}\langle\xi'\rangle^{\alpha_2}\langle\varepsilon\xi'\rangle^{\alpha_3}(\langle\xi\rangle^{-1}\langle\xi'\rangle+\langle\varepsilon\xi\rangle^{-1}\langle\varepsilon\xi'\rangle).$$

The Poisson operator with symbol k is given by

$$\pi_+ \mathrm{Op}_{\xi_n} k(\varepsilon,\xi',\xi_n)\psi(\varepsilon,\xi') = \psi(\varepsilon,\xi').F^{-1}_{\xi_n\to x_n} k(\varepsilon,\xi',\xi_n).$$

In the proofs of the following lemmas, several techniques introduced in [Fr 1] will be heavily used. In particular, the rewriting of certain contour integrals as the sum of two integrals over the contours $\langle\xi'\rangle\Gamma_1$ and $\varepsilon^{-1}\Gamma_2$ with Γ_1,Γ_2 independent upon ε,ξ', will play a fundamental role.

Let L_{01}^+, L_{02}^+ have the form (3.1.19), (3.1.20), with $r_2, r_3 \geq 0$, $a \in L_{\mu^{(1)}}(\mathbb{R}^{n-1})$, let $b \in L_{\mu^{(2)}}(\mathbb{R}^n)$ be a polynomial in ξ_n with $\mu_2^{(2)}+\mu_3^{(2)} \leq r_2+r_3-1$, and let $L_0^+ = L_{01}^+ L_{02}^+$.

<u>Lemma 3.1.25.</u>

Let $(L_0^+(\varepsilon,\xi))^{-1} \in \mathcal{D}^{k_0}_{(0,-r_2,-r_3)}$ and let

$$(3.1.25)\quad k(\varepsilon,\xi) = \frac{a(\varepsilon,\xi')b(\varepsilon,\xi)}{L_0^+(\varepsilon,\xi)}$$

be a Poisson symbol of order $\alpha = \mu^{(1)}+\mu^{(2)}-(0,r_2,r_3)$. Then

$$||\pi_+ \mathrm{Op}_{\xi_n} k(\varepsilon,\widehat{\xi'},\xi_n)\psi||^+_{(s),\xi'} \leq C[\psi]_{(\tau),\xi'} \text{ for } s_2+\tfrac{1}{2}+\mu_2^{(2)}-r_2 < k_0$$
$$||\pi_+ \mathrm{Op}_{\xi_n} k(\varepsilon,\widehat{\xi'},\xi_n)\psi||^+_{(s),\xi'} \leq C[\psi]_{(\sigma),\xi'} \text{ for } s_2+\tfrac{1}{2}+\mu_2^{(2)}-r_2 \geq k_0$$

with a constant C independent upon ε,ξ',ψ, and with

$$\tau = s+\alpha+\tfrac{1}{2}e_2, \qquad e_2 = (0,1,0)$$
$$\sigma = \tau+(s_2+\tfrac{1}{2}+\mu_2^{(2)}-r_2-k_0)e, \qquad e = (1,-1,1).$$

<u>Proof</u>. Following [Fr 1], let $\rho_1 > 0$ and the curves Γ_1,Γ_2 be as in Lemma 3.1.14(ii).

For $\rho = \varepsilon\langle\xi'\rangle \leq \rho_1$, one obtains, introducing the integration variables $t = \langle\xi'\rangle^{-1}\xi_n$, $t = \varepsilon\xi_n$:

$$\pi_+\mathrm{Op}_{\xi_n} k(\varepsilon,\widehat{\xi'},\xi_n)\psi = \psi\cdot\Big(\int_{\Gamma_1} e^{ix_n\langle\xi'\rangle t} k(\varepsilon,\widehat{\xi'},\langle\xi'\rangle t)\langle\xi\rangle dt + \int_{\Gamma_2} e^{ix_n\varepsilon^{-1}t} k(\varepsilon,\widehat{\xi'},\varepsilon^{-1}t)\varepsilon^{-1}dt\Big).$$

For $k_0 = 0$, one has

$$||\pi_+ Op_{\xi_n} k(\varepsilon,\widehat{\xi'},\xi_n)\psi||^+_{(s),\xi'} = ||\Pi^+_{\xi_n}(-i\xi_n+\langle\xi'\rangle)^{s_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3}k(\varepsilon,\widehat{\xi'},\xi_n)\psi||_{L^2(\mathbb{R}_{\xi_n})}$$

$$\le \varepsilon^{-s_1}||\int_{\Gamma_1}(t\langle\xi'\rangle+\langle\xi'\rangle)^{s_2}(\varepsilon t\langle\xi'\rangle+\langle\varepsilon\xi'\rangle)^{s_3}e^{ix_n\langle\xi'\rangle t}k(\varepsilon,\widehat{\xi'},\langle\xi'\rangle t)\langle\xi'\rangle dt$$

$$+\int_{\Gamma_2}(\varepsilon^{-1}t+\langle\xi'\rangle)^{s_2}(t+\langle\varepsilon\xi'\rangle)^{s_3}e^{ix_n\varepsilon^{-1}t}k(\varepsilon,\widehat{\xi'},\varepsilon^{-1}t)\varepsilon^{-1}dt||_{L^2(\mathbb{R}_+(x_n))}|\psi|$$

$$\le C\varepsilon^{-s_1-\alpha_1}(\langle\xi'\rangle^{s_2+\frac12+\mu_2^{(2)}-r_2}+\varepsilon^{-(s_2+\frac12+\mu_2^{(2)}-r_2)})\langle\xi'\rangle^{\mu_2^{(1)}}|\psi|$$

$$\le \begin{cases} C[\psi]_{(\tau),\xi'} & \text{if } s_2+\frac12+\mu_2^{(2)}-r_2 < 0 \\ C[\psi]_{(\sigma),\xi'} & \text{if } s_2+\frac12+\mu_2^{(2)}-r_2 \ge 0. \end{cases}$$

Let now $k_0 > 0$. Then Lemma 3.1.23 implies that $r_3 = 0$, so that $L^+_{02} = 1+r(\varepsilon,\xi)$, where r satisfies (3.1.22). Since $t \to L^+_{01}(\varepsilon,\widehat{\xi'},\varepsilon^{-1}t)$ has no zeroes inside Γ_2, one has

$$||\pi_+ Op_{\xi_n} k(\varepsilon,\widehat{\xi'},\xi_n)\psi||^+_{(s),\xi'} =$$

$$\le C\varepsilon^{-s_1}||\int_{\Gamma_1}(t\langle\xi'\rangle+\langle\xi'\rangle)^{s_2}(\varepsilon t\langle\xi'\rangle+\langle\varepsilon\xi'\rangle)^{s_3}e^{ix_n\langle\xi'\rangle t}k(\varepsilon,\widehat{\xi'},\langle\xi'\rangle t)\langle\xi'\rangle dt -$$

$$-\int_{\Gamma_1}(\varepsilon^{-1}t+\langle\xi'\rangle)^{s_2}(t+\langle\varepsilon\xi'\rangle)^{s_3}e^{ix_n\varepsilon^{-1}t}(abL^{-1}_{01})(\varepsilon,\widehat{\xi'},\varepsilon^{-1}t)r(\varepsilon,\xi',\varepsilon^{-1}t)\varepsilon^{-1}dt|||\psi|$$

$$\le C\varepsilon^{-\alpha_1}((\langle\xi'\rangle)^{s_2+\frac12+\mu_2^{(2)}-r_2-k_0}+\varepsilon^{-(s_2+\frac12+\mu_2^{(2)}-r_2-k_0)})\langle\xi'\rangle^{k_0+\mu_2^{(1)}}|\psi|$$

$$\le \begin{cases} C[\psi]_{(\tau),\xi'} & \text{if } s_2+\frac12+\mu_2^{(2)}-r_2 < k_0 \\ C[\psi]_{(\sigma),\xi'} & \text{if } s_2+\frac12+\mu_2^{(2)}-r_2 \ge k_0. \end{cases}$$

Let now $\varepsilon\langle\xi'\rangle \ge \rho_1$, where $\rho_1 > 0$ is the constant fixed above. Following again [Fr 1], let $\Gamma = \Gamma(\rho_1)$ be the curve constructed in the proof of Lemma 3.1.14 (iii).

Then, introducing the integration variable $t = \langle\xi'\rangle^{-1}\xi_n$, one obtains

$$Op\, k(\varepsilon,\widehat{\xi'},\xi_n)\psi = \frac{1}{2\pi i}\int_\Gamma e^{-ix_n\langle\xi'\rangle t}k(\varepsilon,\widehat{\xi'},\langle\xi'\rangle t)\langle\xi'\rangle dt\ \psi$$

and, therefore,

$$||\pi_+ \mathrm{Op}\, k(\varepsilon,\widehat{\xi'},\xi_n)\psi||^+_{(s),\xi'} \le$$

$$\le C\varepsilon^{-s_1}||\int_\Gamma (t<\xi'>+<\xi'>)^{s_2}(\varepsilon t<\xi'>+<\varepsilon\xi'>)^{s_3} e^{ix_n<\xi'>t} k(\varepsilon,\widehat{\xi'},<\xi'>t)<\xi'>dt||\cdot|\psi|$$

$$\le C\varepsilon^{-(s_1+\alpha_1)}<\xi'>^{s_2+\alpha_2+\frac{1}{2}}<\varepsilon\xi'>^{s_3+\alpha_3}|\psi|$$

$$\le C[\psi]_{(\tau),\xi'}.$$

Since for $\varepsilon<\xi'> \ge \rho_1$ and $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 \le 1$, there is the equivalency

$$<\xi'>^2 \le \varepsilon^{-2}<\varepsilon\xi'>^2 \le (1+\rho_1^{-2})<\xi'>^2,$$

one has

$$[\psi]_{(\sigma),\xi'} \le C[\psi]_{(\tau),\xi'}. \qquad \square$$

Definition 3.1.26.

A rational function $\lambda \to t(\varepsilon,\xi',\lambda)$, whose singularities are located in the half plane $\{\mathrm{Im}\,\lambda < 0\}$, is called a trace symbol of order $\mu \in \mathbb{R}^3$ if there exist numbers α_2,α_3, such that

$$(3.1.26)\quad |t(\varepsilon,\xi',\xi_n)| \le C\varepsilon^{-\mu_1}<\xi>^{\mu_2-\alpha_2}<\varepsilon\xi>^{\mu_3-\alpha_3}<\xi'>^{\alpha_2}<\varepsilon\xi'>^{\alpha_3}$$

$$\forall(\varepsilon,\xi) \in (0,\varepsilon_0] \times \mathbb{R}^n$$

with a constant C.

The trace operator with the symbol $t(\varepsilon,\xi)$ is defined by $\pi_0 t(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})l_0$, where l_0 is the extension operator by zero.

Lemma 3.1.27.

For every trace symbol t of order μ, one has:

$$(3.1.27)\quad [\pi_0 t(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})l_0 u]_{(\tau),\xi'} \le C||u||^+_{(s),\xi'}$$

$$\forall u \in H_{(s),\xi'}(\mathbb{R}_+) \text{ if } s_2-\mu_2+\alpha_2 > \tfrac{1}{2}$$

$$(3.1.28)\quad [\pi_0 t(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 u]_{(\sigma),\xi'} \le C||u||^+_{(s),\xi'}$$

$$\forall u \in H_{(s),\xi'}(\mathbb{R}_+) \quad \text{if } s_2-\mu_2+\alpha_2 < \tfrac{1}{2}$$

with a constant C independent upon ε,ξ',u and with

$$\tau = s-\mu-\tfrac{1}{2}e_2$$

$$\sigma = \tau+(s_2-\mu_2+\alpha_2-\tfrac{1}{2})e.$$

Proof. Since the singularities of $t(\varepsilon,\xi',\circ)$ are located in the half plane $\{\mathrm{Im}\,\lambda < 0\}$, one has:

$$<\xi'>^{-\alpha_2}<\varepsilon\xi'>^{-\alpha_3}t(\varepsilon,\xi'-i\frac{\partial}{\partial x_n})1_0 \in \mathrm{Hom}(H_{(s),\xi'}(\mathbb{R}_+),$$

$$H_{(s-(\mu_1,\mu_2-\alpha_2,\mu_3-\alpha_3)),\xi'}(\mathbb{R}_+))$$

for $\forall s \in \mathbb{R}^3$. The application of the trace theorem established in [Fr 1] and formulated here as Theorem 2.2.4 yields the claim (3.1.27), (3.1.28).

□

Definition 3.1.28.

A function $g(\varepsilon,\xi',\xi_n,\eta_n) = \Sigma k_i(\varepsilon,\xi',\xi_n)t_i(\varepsilon,\xi',\eta_n)$ is called singular Green symbol if the functions k_i and t_i are Poisson and trace symbols, respectively. The corresponding operator is defined by $\pi_0 \mathrm{Op}\, g\, 1_0 =$
$= \pi_+\Sigma k_i(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\pi_0 t_i(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})1_0$, where 1_0 is the extension by zero.

Now we are going to define an algebra of singularly perturbed Wiener-Hopf operators, which is an extension of the operator algebra without small parameter, introduced in [B.d.M.].

Let p be a pseudodifferential symbol, g a singular Green symbol, t_i, $1 \le i \le s$, trace symbols, k_j, $1 \le j \le r$, Poisson symbols and q_{ij}, $1 \le i \le s$, $1 \le j \le r$, pseudodifferential symbols on $\mathbb{R}^{n-1}_{\xi'}$.

Definition 3.1.29.

The matrix

$$(3.1.29)\quad R = \begin{pmatrix} p(\varepsilon,\xi)+g(\varepsilon,\xi',\xi_n,\eta_n) & k_1(\varepsilon,\xi) & \dots & k_r(\varepsilon,\xi) \\ t_1(\varepsilon,\xi) & q_{11}(\varepsilon,\xi') & \dots & q_{1r}(\varepsilon,\xi') \\ \vdots & \vdots & \dots & \vdots \\ t_s(\varepsilon,\xi) & q_{s1}(\varepsilon,\xi') & & q_{sr}(\varepsilon,\xi') \end{pmatrix}$$

is called singularly perturbed Wiener-Hopf symbol and the operator

$$(3.1.30)\quad \mathrm{Op}\, R = \begin{pmatrix} \pi_+\mathrm{Op}(p(\varepsilon,\xi)+g(\varepsilon,\xi,\eta_n))1_0 & \pi_+\mathrm{Op}\; k_1(\varepsilon,\xi) & \dots & \pi_+\mathrm{Op}\; k_r(\varepsilon,\xi) \\ \pi_0\mathrm{Op}\; t_1(\varepsilon,\xi)1_0 & q_{11}(\varepsilon,\xi') & \dots & q_{1r}(\varepsilon,\xi') \\ \vdots & \vdots & & \vdots \\ \pi_0\mathrm{Op}\; t_s(\varepsilon,\xi)1_0 & q_{s1}(\varepsilon,\xi') & \dots & q_{sr}(\varepsilon,\xi') \end{pmatrix}$$

is called the singularly perturbed Wiener-Hopf operator associated with the symbol (3.1.29).

For $1 \le 1 \le 2$, let

$$(3.1.31)\quad R^{(1)} = \begin{pmatrix} p^{(1)}+g^{(1)} & k_1^{(1)} & \dots & k_{r^{(1)}}^{(1)} \\ t_1^{(1)} & q_{11}^{(1)} & \dots & q_{1r^{(1)}}^{(1)} \\ \vdots & \vdots & & \vdots \\ t_{s^{(1)}}^{(1)} & q_{s^{(1)}1}^{(1)} & \dots & q_{s^{(1)}r^{(1)}}^{(1)} \end{pmatrix}$$

be singularly perturbed Wiener-Hopf symbols.

We consider $\mathrm{Op}\, R^{(1)}$ as an operator from the space $\omega_1^{(1)} = L^2(\mathbb{R}_+) \times \prod_{j=1}^{r^{(1)}} \mathbb{C}$ into the space $\omega_2^{(1)} = L^2(\mathbb{R}_+) \times \prod_{i=1}^{s^{(1)}} \mathbb{C}$.

Following [B.d.M.], the product of the symbols $R^{(1)}, R^{(2)}$ will be introduced:

Definition 3.1.30.

If $s^{(2)} = r^{(1)}$, the product $R^{(1)} \circ R^{(2)}$ of the symbols $R^{(1)}$ and $R^{(2)}$ is the symbol $R^{(3)}$ given by (3.1.31), where $s^{(3)} = s^{(1)}$, $r^{(3)} = r^{(1)}$ and where $p^{(3)}, g^{(3)}, k_j^{(3)}, t_i^{(3)}, q_{ij}^{(3)}$ are given by the formulae

$$p^{(3)}(\varepsilon,\xi) = p^{(1)}(\varepsilon,\xi)\cdot p^{(2)}(\varepsilon,\xi)$$

$$\begin{aligned} g^{(3)}(\varepsilon,\xi,\eta_n) = {} & \Pi^+_{\xi_n}\Pi^-_{\eta_n}[(\Pi^+p^{(1)}(\varepsilon,\xi)-\Pi^+p^{(1)}(\varepsilon,\xi',\eta_n))\cdot \\ & \cdot(\Pi^-p^{(2)}(\varepsilon,\xi)-\Pi^-p^{(2)}(\varepsilon,\xi',\eta_n))(i\xi_n-i\eta_n)^{-1}] \\ & + \Pi^+_{\xi_n}p^{(1)}(\varepsilon,\xi)g^{(2)}(\varepsilon,\xi',\xi_n,\eta_n) + \\ & + \Pi^-_{\eta_n}g^{(1)}(\varepsilon,\xi',\xi_n,\eta_n)p^{(2)}(\varepsilon,\xi',\eta_n) \\ & + \int_\gamma g^{(1)}(\varepsilon,\xi\ ,\zeta_n)g^{(2)}(\varepsilon,\xi',\zeta_n,\eta_n)d\zeta_n \\ & + \sum_{1\le j\le r^{(1)}} k_j^{(1)}(\varepsilon,\xi',\xi_n)t_j^{(2)}(\varepsilon,\xi',\eta_n) \end{aligned}$$

$$\begin{aligned} k_j^{(3)}(\varepsilon,\xi) = {} & \Pi^+_{\xi_n}p^{(1)}(\varepsilon,\xi)k_j^{(2)}(\varepsilon,\xi)+\int_\gamma g^{(1)}(\varepsilon,\xi',\eta_n)k^{(2)}(\varepsilon,\xi',\eta_n)d\eta_n+ \\ & + \sum_{1\le i\le r^{(1)}} k_i^{(1)}(\varepsilon,\xi)q_{ij}^{(2)}(\varepsilon,\xi'), \qquad 1 \le j \le r^{(3)} \end{aligned}$$

$$\begin{aligned} t_i^{(3)}(\varepsilon,\xi) = {} & \Pi^-_{\xi_n}t_i^{(1)}(\varepsilon,\xi)p^{(2)}(\varepsilon,\xi)+\int_\gamma t_i^{(1)}(\varepsilon,\xi',\eta_n)g^{(2)}(\varepsilon,\xi',\eta_n,\xi_n)d\eta_n \\ & + \sum_{1\le j\le s^{(2)}} q_{ij}^{(1)}(\varepsilon,\xi')t_j^{(2)}(\varepsilon,\xi), \qquad 1 \le i \le s^{(3)} \end{aligned}$$

$$\begin{aligned} q_{ij}^{(3)}(\varepsilon,\xi') = {} & \int_\gamma t_i^{(1)}(\varepsilon,\xi)k_j^{(2)}(\varepsilon,\xi)d\xi_n + \sum_{1\le l\le s^{(2)}} q_{il}^{(1)}(\varepsilon,\xi')q_{lj}^{(2)}(\varepsilon,\xi') \\ & 1 \le i \le s^{(3)},\ 1 \le j \le r^{(3)} \end{aligned}$$

where $\gamma = \gamma(\varepsilon,\xi')$ is a closed Jordan curve in the upper half plane which encloses all the singularities of the integrand.

The set of singularly perturbed Wiener-Hopf symbols will be denoted by X and the set of corresponding operators by Op X.

The following result is analogous to the one established for Wiener-Hopf operators without small parameter in [B.d.M.]:

Lemma 3.1.31.

Let $R^{(j)} \in X$, $j = 1,2$. Then

(i) $\mathrm{Op}\, R^{(1)} + \mathrm{Op}\, R^{(2)} \in \mathrm{Op}\, X$ if $R^{(1)}+R^{(2)} \in X$ is well defined ($r^{(1)} = r^{(2)}$, $s^{(1)} = s^{(2)}$)

(ii) $\mathrm{Op}\, R^{(1)} \circ \mathrm{Op}\, R^{(2)} \in \mathrm{Op}\, X$ if $R^1 R^2 \in X$ is well defined ($s^{(2)} = r^{(1)}$)

(iii) $\mathrm{Op}({}^tR^{(1)}) \in \mathrm{Op}\, X$, where ${}^tR^{(1)}$ is the adjoint operator to $\mathrm{Op}\, R^{(1)}$ with respect to the scalar products in $\omega_1^{(1)}, \omega_2^{(1)}$.

Proof. (i) According to the definitions, the sum of two singularly perturbed pseudodifferential symbols (or singular Green symbols, Poisson symbols and trace symbols) is again such a symbol. This proves the first part of Lemma 3.1.31.

(ii) Let now $p \in L_\nu$ be a pseudodifferential symbol. Following [Fr 1], let $\rho_1 > 0$ and the curves Γ_1, Γ_2 be as in Lemma 3.1.14(ii). With $\gamma = \gamma(\varepsilon,\xi')$ a curve in the half plane which encloses all singularities of the function $\eta_n \to p(\varepsilon,\xi',\eta_n)$, one obtains for $\rho = \varepsilon\langle\xi'\rangle \le \rho_1$:

$$\Pi^+_{\xi_n} p(\varepsilon,\xi) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\xi_n-\eta_n}\, p(\varepsilon,\xi',\eta_n)\,d\eta_n$$

$$= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\langle\xi'\rangle}{\xi_n-\langle\xi'\rangle t} p(\varepsilon,\xi',\langle\xi'\rangle t)\,dt + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\varepsilon^{-1}}{\xi_n-\varepsilon^{-1}t} p(\varepsilon,\xi',\varepsilon^{-1}t)\,dt$$

since the curves $\langle\xi'\rangle\Gamma_1$ and $\varepsilon^{-1}\Gamma_2$ are disjoint.

Hence,

$$|\Pi^+_{\xi_n} p(\varepsilon,\xi)| \le C\varepsilon^{-\nu_1}\Big(\frac{\langle\xi'\rangle}{\langle\xi\rangle}\langle\xi'\rangle^{\nu_2}\langle\varepsilon\xi'\rangle^{\nu_3} + \frac{\langle\varepsilon\xi'\rangle}{\langle\varepsilon\xi\rangle}\langle(\xi',\varepsilon^{-1})\rangle^{\nu_2}\langle(\varepsilon\,\xi',t)\rangle^{\nu_3}\Big)$$

$$\le \begin{cases} C\varepsilon^{-\nu_1-\nu_2}\langle\varepsilon\xi'\rangle^{\nu_2+\nu_3}(\langle\xi\rangle^{-1}\langle\xi'\rangle+\langle\varepsilon\xi\rangle^{-1}\langle\varepsilon\xi'\rangle) & \text{if } \nu_2 \ge 0 \\ C\varepsilon^{-\nu_1}\langle\xi'\rangle^{\nu_2}\langle\varepsilon\xi'\rangle^{\nu_3}(\langle\xi\rangle^{-1}\langle\xi'\rangle+\langle\varepsilon\xi\rangle^{-1}\langle\varepsilon\xi'\rangle) & \text{if } \nu_2 \le 0. \end{cases}$$

Thus, for $p \in L_\nu$, Π^+p is a Poisson symbol and Π^-p is a trace symbol. Therefore, with $\mathcal{R}^{(j)} = \mathrm{Op}\, R^{(j)}$, the functions $g^{(3)}$, $k_j^{(3)}$, $t_i^{(3)}$, $q_{ij}^{(3)}$ introduced in Definition 3.1.30 are singular Green, Poisson, trace and pseudodifferential symbols, respectively, and $R^{(1)} \circ R^{(2)} = \mathrm{Op}\, R^{(3)} \in \mathrm{Op}\, X$.

(iii) The scalar products in $\omega_1^{(1)}, \omega_2^{(1)}$ are given by

$$(U^{(1)},U^{(2)})_{\omega_1^{(1)}} = (u^{(1)},u^{(2)})_{L^2(\mathbb{R}_+)} + \sum_{j=1}^{r^{(1)}} \psi_j^{(1)}\overline{\psi_j^{(2)}}$$

for $U^{(l)} = (u^{(l)},\psi_1^{(l)},\ldots,\psi_{r^{(1)}}^{(l)}) \in \omega_1^{(1)}$, $l = 1,2$

and by

$$(F^{(1)},F^{(2)})_{\omega_2^{(1)}} = (f^{(1)},f^{(2)})_{L^2(\mathbb{R}_+)} + \sum_{j=1}^{s^{(1)}} \phi_j^{(1)}\overline{\phi_j^{(2)}}$$

for $F^{(l)} = (f^{(l)},\phi_1^{(l)},\ldots,\phi_{s^{(1)}}^{(l)}) \in \omega_2^{(1)}$, $l = 1,2$.

As a consequence of Parseval's identity, one has

$$\phi\, \overline{\pi_0 \mathrm{Op}\, t(\varepsilon,\xi)\, l_0 u} = (\pi_+ \mathrm{Op}\, \overline{t(\varepsilon,\xi',-\xi_n)}\phi,u)_{L^2(\mathbb{R}_+)} \quad \forall\phi \in \mathbb{C},\ \forall u \in L^2(\mathbb{R}_+)$$

for trace symbols t. Hence,

$$(F,\mathrm{Op}\, R^{(1)}U)_{\omega_2^{(1)}} = (\mathrm{Op}\, R^{(2)}F,U)_{\omega_1^{(1)}} \quad \forall U \in \omega_1^{(1)},\ \forall F \in \omega_2^{(1)}$$

where $R^{(2)}$ has the form (3.1.31) with $r^{(2)} = s^{(1)}$, $s^{(2)} = r^{(1)}$ and with $p^{(2)},g^{(2)},k_i^{(2)},t_j^{(2)},q_{ij}^{(2)}$ given by the formulae

$$p^{(2)}(\varepsilon,\xi) = \overline{p^{(1)}(\varepsilon,\xi)}$$

$$g^{(2)}(\varepsilon,\xi',\xi_n,\eta_n) = \overline{g^{(1)}(\varepsilon,\xi',-\eta_n,-\xi_n)}$$

$$k_i^{(2)}(\varepsilon,\xi) = \overline{t_i^{(1)}(\varepsilon,\xi',-\xi_n)}, \qquad 1 \le i \le r^{(2)}$$

$$t_j^{(2)}(\varepsilon,\xi) = \overline{k_j^{(1)}(\varepsilon,\xi',-\xi_n)}, \qquad 1 \le j \le s^{(2)}$$

$$q_{ji}^{(2)}(\varepsilon,\xi') = \overline{q_{ij}^{(1)}(\varepsilon,\xi')}, \qquad 1 \le j \le s^{(2)},\ 1 \le i \le r^{(2)}. \quad \square$$

The following result, which was established in [Fr 1], will be needed later for the proof of basic estimate.

Let $A_{kj}(\rho,\omega') \in \mathrm{Hom}(\mathbb{C}^{r_{j+1}},\mathbb{C}^{r_{k+1}})$, $1 \le k,\ j \le 2$, be matrices which depend continuously upon the parameters $(\rho,\omega') \in [0,\rho_1] \times \Omega_{n-1}$. Let $T_{\rho,\omega'}$ be a block matrix of the form (with $\alpha,\beta \in \mathbb{R}$):

$$T_{\rho,\omega'} = \begin{pmatrix} Id_{r_2}, & \rho^{-\alpha} A_{12}(\rho,\omega') \\ \rho^{\beta} A_{21}(\rho,\omega'), & Id_{r_3} \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^{r_2+r_3},\mathbb{C}^{r_2+r_3})$$

where $Id_{r_k} \in \mathrm{Hom}(\mathbb{C}^{r_k},\mathbb{C}^{r_k})$ is the identity matrix. One has:

<u>Lemma 3.1.31.</u> ([Fr 1])

If $\alpha < \beta$, then there is $\rho_0 > 0$, such that for $\forall\rho \in (0,\rho_0]$, there exists the inverse $T^{-1}_{\rho,\omega'}$ of the matrix $T_{\rho,\omega'}$ which has the form:

$$T^{-1}_{\rho,\omega'} = Id + \rho^{\beta-\alpha}\begin{pmatrix} B_{11}(\rho,\omega') & 0 \\ 0 & B_{22}(\rho,\omega') \end{pmatrix} + \begin{pmatrix} 0 & \rho^{-\alpha}B_{12}(\rho,\omega') \\ \rho^{\beta}B_{21}(\rho,\omega') & 0 \end{pmatrix},$$

where $B_{kj}(\rho,\omega') \in \mathrm{Hom}(\mathbb{C}^{r_{j+1}},\mathbb{C}^{r_{k+1}})$, $1 \le k, j \le 2$, are continuous functions of $(\rho,\omega') \in [0,\rho_0] \times \Omega_{n-1}$.

<u>Proof</u>. See the proof of Lemma 3.2.3 in [Fr 1]. $\square$

3.2 The a priori estimate

Let $Q_0^{(1)}(\varepsilon,\xi)$ and $Q_0^{(2)}(\varepsilon,\xi)$ be two properly elliptic principal symbols of orders $\nu^{(1)}$ and $\nu^{(2)}$, respectively. Let $\nu_j^{(k)} = 2r_j^{(k)}$,

$$k = 1,2\ ,\ j = 2,3, \text{ with } r_j^{(k)} \text{ nonnegative integers.}$$

The rational function

$$L_0(\varepsilon,\xi) = Q_0^{(1)}(\varepsilon,\xi) \quad (Q_0^{(2)}(\varepsilon,\xi))^{-1}$$

is elliptic of order $\nu = \nu^{(1)}-\nu^{(2)}$. Moreover, we call the vector $r = (0,\ r_2^{(1)}-r_2^{(2)},\ r_3^{(1)}-r_3^{(2)})$ the index of the rational function L_0. It will be assumed that L_0 belongs to the class $D_\nu^{k_0}$ and that the integers r_2, r_3 are nonnegative:

$$(3.2.1) \qquad r_j \geq 0,\quad j = 2,\ 3\ .$$

Let $L_{j0}(\varepsilon,\xi)$, $1 \leq j \leq r_2 + r_3$, be trace symbols with vectorial orders

$$\mu_j = (\gamma_j,m_j,p_j) \in \mathbb{R} \times Z \times Z,\quad 1 \leq j \leq r_2 + r_3$$

which are supposed to be homogeneous in (ε^{-1},ξ) of order γ_j+m_j.

With π_+ and π_0 being the restriction operators to the half line $\mathbb{R}_+$ and to the point $x_n = 0$, respectively, and with

$$\widehat{\xi'} = \langle\xi'\rangle\omega',\ \omega' = \ \xi'\ |\xi'|^{-1},\ \xi' \in \mathbb{R}^{n-1} \setminus \{0\}$$

consider the following singularly perturbed problem in $\mathbb{R}_+$ with the parameters $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$:

$$(3.2.2) \qquad \pi_+ L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})\, u_+(x_n) = f(x_n),\quad x_n > 0$$

$$(3.2.3) \qquad \pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})\ u_+ \ = \phi_j,\quad 1 \leq j \leq r_2+r_3.$$

Here the solutionu_+ has its support in $\overline{\mathbb{R}}_+$ and $\pi_+u_+ \in H_{(s),\xi'}(\mathbb{R}_+)$, where $s \in \mathbb{R}^3$ will be chosen later on. Without loss of generality, it is assumed that the boundary operators L_{j0} are ordered in such a way that

$$m_1 \le \dots \le m_{r_2} \le m_{r_2+1} \le \dots \le m_{r_2+r_3}.$$

The rational function $\lambda \to L_0(\varepsilon,\xi',\lambda)$ can, according to lemma 3.1.13, be factorized in the following way:

$$(3.2.4) \qquad L_0(\varepsilon,\xi',\lambda) = L_0^+(\varepsilon,\xi',\lambda)\, L_0^-(\varepsilon,\xi',\lambda)$$

where the functions

$$(3.2.5) \qquad L_0^+(\varepsilon,\xi',\lambda) = Q_0^{(1)+}(\varepsilon,\xi',\lambda)\ (Q_0^{(2)+}(\varepsilon,\xi',\lambda))^{-1}$$

and

$$(3.2.6) \qquad L_0^-(\varepsilon,\xi',\lambda) = Q_0^{(1)-}(\varepsilon,\xi',\lambda)\ (Q_0^{(2)-}(\varepsilon,\xi',\lambda))^{-1}$$

have their zeroes and singularities in the half planes $\{\mathrm{Im}\,\lambda > 0\}$ and $\{\mathrm{Im}\,\lambda < 0\}$, respectively. Moreover, L_0^+ satisfies the following inequality:

$$\begin{aligned} C^{-1}(|\xi'|+|\lambda|)^{r_2}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} &\le |L_0^+(\varepsilon,\xi',\lambda)| \le \\ &\le C(|\xi'|+|\lambda|)^{r_2}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} \end{aligned}$$

when $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $\mathrm{Im}\,\lambda \le 0$, $\varepsilon \in (0,\varepsilon_0]$.

The function $\lambda \to L_0^+(\varepsilon,\xi',\lambda)$ can for ε_0 sufficiently small be factorized as follows:

$$L_0^+(\varepsilon,\xi',\lambda) = L_{01}^+(\varepsilon,\xi',\lambda)\ L_{02}^+(\varepsilon,\xi',\lambda)$$

with L_{01}^+ homogeneous in $(\varepsilon^{-1},\xi',\lambda)$ of order r_2 and satisfying the inequalities:

$$C^{-1}(|\xi'|+|\lambda|)^{r_2} \le |L^+_{01}(\varepsilon,\xi',\lambda)| \le C(|\xi'|+|\lambda|)^{r_2}$$

$\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $\forall\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$, and L^+_{02} homogeneous in $(\varepsilon^{-1},\xi',\lambda)$ of order 0 and satisfying the inequalities:

$$C^{-1}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} \le |L^+_{02}(\varepsilon,\xi',\lambda)| \le C(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3}$$

$\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $\forall\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$.

The representation (3.2.5) implies the existence of the limit

$$(3.2.7) \qquad \lim_{\rho\downarrow 0} L^+_{01}(\rho,\xi',\lambda) = L^{0+}_0(\xi',\lambda)$$

where L^{0+}_0 is a homogeneous function in (ξ',λ) of order r_2. Let $L^+(\lambda)$ be defined by

$$(3.2.8) \qquad L^+(\lambda) = L^+_{02}(1,0,\lambda).$$

Following [Fr 1], let $\rho_1 > 0$ and the curves Γ_1, Γ_2 be as in Lemma 3.1.14(ii). Then Γ_1 and Γ_2 contain all the zeroes of the rational functions

$$\lambda \to L^{0+}_0(\omega',\lambda), \quad \omega' \in \Omega_{n-1},$$

and

$$\lambda \to L^+(\lambda),$$

respectively. Moreover, let L^+_{00} be defined as follows:

$$(3.2.9) \qquad L^+_{00}(\xi',\lambda) = \lim_{\rho\to\infty} \rho^{-r_3} L^+_0(\rho,\xi',\lambda).$$

L_{00}^{+} is homogeneous in (ξ',λ) of order r_2+r_3. Let Γ_3 be a compact Jordan curve in the half plane $\{\mathrm{Im}\,\lambda > 0\}$ which encloses all the zeroes of the functions $L_{00}^{+}(\omega',\lambda)$ for $\omega' \in \Omega_{n-1}$.

First, it will be assumed that the numbers m_j, r_2 satisfy the condition

$$(3.2.10) \qquad \max(r_2-1,\ m_{r_2}) < \min(k_0,\ m_{r_2+1}).$$

Then introduce l_0, l_1, α, β by

$$(3.2.11) \qquad \begin{cases} l_0 \overset{\text{def}}{=} r_2 \\ l_1 \overset{\text{def}}{=} r_2+r_3 \\ \alpha \overset{\text{def}}{=} \max(r_2-1,\ m_{r_2}) \\ \beta \overset{\text{def}}{=} \min(k_0,\ m_{r_2+1}) \end{cases}$$

In the differential case, one can take $k_0 = \infty$ according to Lemma 3.1.19. Moreover, if in the differential case the reduced problem is coercive, one has $m_{r_2} \geq r_2-1$. Indeed, $m_1 \leq \ldots \leq m_{r_2} \leq r_2-2$ would imply that the polynomials $L_{j0}^{0}(\xi) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma_j} L_{j0}(\varepsilon,\xi)$, $1 \leq j \leq r_2$, are linearly dependent modulo $L_0^{0+}(\xi)$. Therefore, in the differential case, (3.2.10) is essentially equivalent to the condition $m_{r_2} < m_{r_2+1}$ (see [Fr]). The reduced problem of (3.2.2), (3.2.3) is given by

$$(3.2.12) \qquad \pi_+ \, L_0^0(\widehat{\xi'}, -i\frac{\partial}{\partial x_n}) \quad u_+(x_n) = f(x_n),\ x_n > 0$$

$$(3.2.13) \qquad \pi_0 \, L_{j0}^0(\widehat{\xi'}, -i\frac{\partial}{\partial x_n}) \qquad u_+ = \phi_j,\ 1 \leq j \leq r_2.$$

The boundary operators L_{j0} are supposed to satisfy the following coerciveness condition, introduced in [Fr 1]:

(i) Let $L_{j0}^{0}(\xi) = \lim_{\varepsilon \downarrow 0} \varepsilon^{\gamma_j} L_{j0}(\varepsilon,\xi)$ be the reduced symbol of L_{j0}. With Γ_1 the curve defined above, which encloses all the zeroes of $\lambda \to L_0^{0+}(\omega',\lambda)$, introduce

$$q_{kj}^{0}(\omega') = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{L_{j0}^{0}(\omega',\lambda)\ \lambda^{k-1}\ d\lambda}{L_0^{0+}(\omega',\lambda)}\ , \ 1 \le k,\ j \le l_0 .$$

The matrix $||q_{kj}^{0}(\omega')||$ is supposed to be non-singular for any $\omega' \in \Omega_{n-1}$:

$$(3.2.14) \qquad \det ||q_{kj}^{0}(\omega')||_{1 \le k, j \le l_0} \neq 0, \quad \forall \omega' \in \Omega_{n-1}.$$

(ii) Let $L_j(\lambda) = L_{j0}(1,0,\lambda)$. With Γ_2 the curve defined above, which encloses all the zeroes of $\lambda \to L^{+}(\lambda)$, introduce

$$(3.2.15) \qquad q_{kj} = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{L_j(\lambda)}{L^{+}(\lambda)} \lambda^{k-r_2-1}\ d\lambda,\ l_0 < k,j \le l_1 .$$

The matrix $||q_{kj}||$ is supposed to be non-singular:

$$(3.2.16) \qquad \det||q_{kj}||_{l_0<k,j\le l_1} \neq 0.$$

(iii) Let $L_{j00}(\xi)$ be the principal homogeneous symbol of order m_j+p_j for $L_{j0}(1,\xi)$, $1\le j\le l_1$, so that

$$L_{j00}(\omega',\lambda) = \lim_{\rho\to\infty} \rho^{\gamma_j-p_j} L_{j0}(\rho,\omega',\lambda).$$

Let Γ_3 be the curve defined above, which encloses all the zeroes and singularities of $\lambda \to L_{00}^{+}(\omega',\lambda)$. With

$$(3.2.17) \qquad q_{kj}^{00}(\omega') = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{L_{j00}(\omega',\lambda)\lambda^{k-1}}{L_{00}^{+}(\omega',\lambda)}\ d\lambda, \quad 1 \le k,j \le l_1,$$

it is assumed that

$$(3.2.18) \qquad \det\|q^{00}_{kj}(\omega')\|_{1\le k,j\le l_1} \neq 0, \ \forall\omega' \in \Omega_{n-1}.$$

(iv) For $\forall(\rho,\omega') \in (0,\infty)\times\Omega_{n-1}$, let $\gamma = \gamma(\rho,\omega')$ be a compact Jordan contour in the upper half plane $\{\mathrm{Im}\,\lambda > 0\}$ which encloses alle the zeroes of $\lambda \to L_0^+(\rho,\omega',\lambda)$. Introduce

$$(3.2.19) \qquad Q_{kj}(\rho,\omega') = \frac{1}{2\pi i} \int_{\gamma(\rho,\omega')} \frac{\rho^{\gamma_j - p_j + r_3} L_{j0}(\rho,\omega',\lambda)\lambda^{k-1}d\lambda}{L_0^+(\rho,\omega',\lambda)}, \quad 1\le k,j\le l_1$$

The matrix $\|Q_{kj}\|$ is supposed to be non-singular:

$$(3.2.20) \qquad \det\|Q_{kj}(\rho,\omega')\|_{1\le k,j\le l_1} \neq 0 \ \forall(\rho,\omega') \in (0,\infty)\times\Omega_{n-1}.$$

Since $\alpha > \beta$, one can choose $s_2, s_3 \in \mathbb{R}$ such that the following conditions are satisfied:

$$(3.2.21) \qquad \alpha + \tfrac{1}{2} < s_2 < \beta + \tfrac{1}{2}$$

$$(3.2.22) \qquad s_2 + s_3 > \max\left(\max_{1\le j\le r_2+r_3}(m_j+p_j),\ r_2+r_3-1\right) + \tfrac{1}{2}.$$
$$s_3 \ge 0$$

The following theorem is in its formulation (and its proof) very similar to Theorem 3.2.11 in [Fr 1]:

<u>Theorem 3.2.1.</u> Assume that the function $L_0(\varepsilon,\xi)$ is an elliptic singularly perturbed rational symbol of order ν, that (3.2.10) holds and that the boundary symbols $L_{j0}(\varepsilon,\xi)$, $1 \le j \le r_2+r_3$, satisfy the coerciveness condition (i) - (iv). Then the problem (3.2.2), (3.2.3) has a unique solution u for $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, which satisfies the following two-sided estimate:

$$(3.2.23)\qquad C^{-1}(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le r_2} [\phi_j]_{(\tau_j),\xi'} + \sum_{r_2<j\le r_2+r_3} [\phi_j]_{(\sigma_j)\xi'}) \le$$

$$\le ||u||^+_{(s),\xi'} \le C(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le r_2} [\phi_j]_{(\tau_j),\xi'} + \sum_{r_2<j\le r_2+r_3} [\phi_j]_{(\sigma_j),\xi'})$$

Here $s = (s_1,s_2,s_3)$ satisfies (3.2.21), (3.2.22), τ_j, σ_j are defined by:

$\tau_j = s - \mu_j - \frac{1}{2}e_2$, $1 \le j \le r_2+r_3$

$\sigma_j = \tau_j + (s_2-m_j-\frac{1}{2})(1,-1,1)$, $1 \le j \le r_2+r_3$

and C is a constant which does not depend upon $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$.

Remark 3.2.2. A two-sided a priori estimate is possible only if, in addition to the condition $m_{r_2}+\frac{1}{2} < s_2 < m_{r_2+1}+\frac{1}{2}$ (see [Fr 1]), one has $s_2 \le k_0+\frac{1}{2}$. Here k_0 is the number up to which the symbol L_0 satisfies the extension of the of the smoothness condition in [V-E] to symbols with small parameter (Definition 3.1.15). The additional restriction on s_2 is needed because pseudodifferential operators of the form $\pi_+L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})l_0$ with $L_0 \in D^{k_0}$ map functions in $C^\infty(\overline{\mathbb{R}_+})$ into functions which contain a boundary layer term of the form $\varepsilon^{k_0} w_\varepsilon(\xi',\varepsilon^{-1}x_n)$ (Lemma 3.1.20).

The proof of Theorem 3.2.1 given below is under many aspects similar to the one of Theorem 3.2.11 in [Fr 1]. In particular, in the case $\varepsilon <\xi'> \le \rho_0$, $f \equiv 0$, the boundary value problem is reduced to the same type of type of algebraic systems on the boundary as in [Fr 1] and the case $\varepsilon <\xi'> \ge \rho_0$, $f \equiv 0$ is treated in a completely analogous way. Besides, the technique of rewriting certain contour integrals as the sum of the two integrals over the contours $<\xi'>\,\Gamma_1$ and $\varepsilon^{-1}\Gamma_2$, where Γ_1, Γ_2 are independent upon ε,ξ' (see [Fr]) is heavily used. This splitting of the solution into a smooth part and a boundary layer is also possible in the pseudodifferential case considered here. However, a separation of L^+_{01} and L^+_{02}, as in formula (3.2.8) in [Fr 1], is, in general, impossible. Moreover, the construction of a special solution in $H_{(s)\xi'}(\mathbb{R}_+)$ to the inhomogeneous problem cannot be immediately extended to the pseudodifferential case. Both difficulties can be overcome, using the results from section 3.1, in particular the factorization in Lemma 3.1.21.

Several auxiliary results will be established in order to prove the Theorem 3.2.1 stated above.

<u>Lemma 3.2.3</u>. The first inequality in (3.2.23) holds with a constant C which does not depend upon $(\varepsilon,\xi') \in (0,\varepsilon_0) \times \mathbb{R}^{n-1} \setminus \{0\}$, $u \in H_{(s),\xi'}(\mathbb{R}_+)$.

<u>Proof</u>. Since $L_{j0}(\varepsilon,\xi',\lambda)$ are -as trace symbols- analytic functions of λ for $\mathrm{Im}\,\lambda > 0$, the condition in Definition 3.1.15 is satisfied for $\forall\, \mu \in \mathbb{R}^3$. Hence, $L_{j0}(\varepsilon,\widehat{\xi'},\xi_n) \in \mathcal{D}^{\infty}_{\mu_j}$ for $1 \le j \le r_2+r_3$ and Lemma 3.1.17 yields

$$||\pi_+ L_{j0}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})1_0 u||^+_{(s-\mu_j),\xi'} \le C||u||^+_{(s),\xi'}\ , \quad 1 \le j \le r_2+r_3.$$

The inequalities (3.2.21) imply that $s_2-m_j > \tfrac{1}{2}$ for $1 \le j \le r_2$ and $s_2-m_j < \tfrac{1}{2}$ for $r_2 < j \le r_2+r_3$. The trace theorem established in [Fr 1] and formulated here as Theorem 2.2.4 yields:

$$[\phi_j]_{(\tau_j),\xi'} = [\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})1_0 u]_{(\tau_j),\xi'} \le C||u||^+_{(s),\xi'},\ 1\le j\le r_2$$

$$[\phi_j]_{(\sigma_j),\xi'} = [\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})1_0 u]_{(\sigma_j),\xi'} \le C||u||^+_{(s),\xi'},\ r_2<j\le r_2+r_3.$$

The conditions $s_2 < k_0+\tfrac{1}{2}$, $s_3 \ge 0$ and Lemma 3.1.17 imply that:

$$||f||^+_{(s-\nu),\xi'} = ||\pi_+ L_0(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})1_0 u||^+_{(s-\nu),\xi'} \le C||u||^+_{(s),\xi'}. \qquad \square$$

In order to prove the second of the inequalities (3.2.23), consider first the case when $f \equiv 0$ and $\rho = \varepsilon \langle\xi'\rangle \le \rho_0$, where ρ_0 is a sufficiently small positive constant which will be chosen later on.

For any polynomial $\xi_n \to M_j(\xi_n)$ of order less than r_2+r_3, the function

$$(3.2.24) \qquad v_j(x_n) = \frac{1}{2\pi i} \int_\gamma \frac{e^{ix_n\lambda} M_j(\lambda)}{L_0^+(\varepsilon,\xi',\lambda)} \, d\lambda, \; x_n > 0$$

is a solution of the equation (3.2.2) with $f \equiv 0$. Here $\gamma = \gamma(\varepsilon,\xi')$ is a Jordan contour in the half plane $\{\operatorname{Im} \lambda > 0\}$ which encloses all the zeroes of the rational function $\lambda \to L_0^+(\varepsilon,\xi',\lambda)$. Since the zeroes and singularities of $\lambda \to L_0^-(\varepsilon,\xi',\lambda)$ are located in the half plane $\{\operatorname{Im} \lambda < 0\}$, $(L_0^-)^{-1}$ satisfies the condition in the definition 3.1.15 for any μ_2 and with $r(\varepsilon,\xi) \equiv 0$. Therefore, one has: $(L_0^-)^{-1} \in \mathcal{D}^\infty_{(-\nu_1,r_2-\nu_2,r_3-\nu_3)}$.
Lemma 3.1.18 yields $L_0^+ = L_0(L_0^-)^{-1} \in \mathcal{D}^{k_0}_{(0,r_2,r_3)}$.
Since the zeroes and singularities of the function $L_{01}^+(\varepsilon,\xi',\lambda)$ grow like $\langle\xi'\rangle$ uniformly with respect to ε, one has $(L_{01}^+)^{-1} \in \mathcal{D}^\infty_{(0,-r_2,0)}$ (see Lemma 3.1.19). Applying Lemma 3.1.18 again, we obtain $L_{02}^+ = L_0^+(L_{01}^+)^{-1} \in \mathcal{D}^{k_0}_{(0,0,r_3)}$. According to Lemma 3.1.21, L_{02}^+ can be decomposed as follows: $L_{02}^+ = c.d$, where $c \in \mathcal{L}_{(0,0,r_3)}$ is a polynomial in λ and where $d \in \mathcal{D}^{k_0}_{(0,0,0)}$.

For $1 \le j \le r_2+r_3$, we choose the polynomials M_j as follows:

$$M_j(\varepsilon,\widehat{\xi'},\lambda) = \begin{cases} \lambda^{j-1} c(\varepsilon,\widehat{\xi'},\lambda) & , \quad 1 \le j \le r_2 \\ \lambda^{j-1} & , \quad r_2 < j \le r_2+r_3 \end{cases}$$

Following [Fr 1], we introduce the number $\rho_1 > 0$ and the curves Γ_1, Γ_2 as in Lemma 3.1.14(ii). For $\rho = \varepsilon\langle\xi'\rangle \in (0,\rho_1]$, $\omega' \in \Omega_{n-1}$, $1 \le k \le r_2+r_3$,

$1 \le j \le r_2$, one has:

(3.2.25) $$\pi_0 L_{k0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 v_j =$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{L_{k0}(\varepsilon,\widehat{\xi'},\lambda)M_j(\varepsilon,\widehat{\xi'},\lambda)}{L_0^+(\varepsilon,\widehat{\xi'},\lambda)}\, d\lambda =$$

$$= \langle\xi'\rangle^{\gamma_k+m_k+j-r_2} \cdot \frac{1}{2\pi i} \int_{\Gamma_1} \frac{L_{k0}(\rho,\omega',t)t^{j-1}c(\rho,\omega',t)}{L_{01}^+(\rho,\omega',t)L_{02}^+(\rho,\omega',t)}\, dt +$$

$$+ \varepsilon^{-(\gamma_k+m_k+j-r_2)} \cdot \frac{1}{2\pi i} \int_{\Gamma_2} \frac{L_{k0}(1,\rho\omega',t)t^{j-1}c(\rho,\omega',t)}{L_{01}^+(1,\rho\omega',t)L_{02}^+(1,\rho\omega',t)}\, dt.$$

As a consequence of Lemma 3.1.22, $d = c^{-1}L_{02}^+ \in \mathcal{D}^{k_0}_{(0,0,0)}$ implies that $d^{-1} = (L_{02}^+)^{-1}c \in \mathcal{D}^{k_0}_{(0,0,0)}$. According to Lemma 3.1.20, d^{-1} can be decomposed as follows: $d^{-1} = 1+r$, where r satisfies (3.1.22).

Since $t \to L_{01}^+(1,\rho\omega',t)$ is analytical inside Γ_2, one has

$$\pi_0 L_{k0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 v_j =$$

$$= \langle\xi'\rangle^{\gamma_k+m_k+j-r_2} \cdot \frac{1}{2\pi i} \int_{\Gamma_1} \frac{L_{k0}(\rho,\omega',t)t^{j-1}}{L_{01}^+(\rho,\omega',t)d(\rho,\omega',t)}\, dt +$$

$$+ \varepsilon^{-(\gamma_k+m_k+j-r_2)} \cdot \frac{1}{2\pi i} \int_{\Gamma_2} \frac{L_{k0}(1,\rho\omega',t)t^{j-1}r(1,\rho\omega',t)}{L_{01}^+(1,\rho\omega',t)}\, dt.$$

According to Lemma 3.1.14 (ii), the zeroes of the functions $t \to L_{02}^+(\rho,\omega',t)$ and $t \to L_{01}^+(1,\rho\omega',t)$ are located outside the curves Γ_1 and Γ_2, respectively. Using this fact and (3.1.22), one obtains that:

(3.2.26) $$\pi_0 \,\mathrm{Op}\, L_{k0}\, 1_0\, v_j = \varepsilon^{-\gamma_k} \langle\xi'\rangle^{m_k+j-r_2} \cdot \delta_1 \quad q^0_{kj}(\rho,\omega') + \varepsilon^{-(\gamma_k+m_k+j-r_2)} \cdot O(\rho^{k_0}), \qquad , 1\le j\le r_2.$$

for $\rho \downarrow 0$.

Here $\delta_1 = (\overset{+}{L}_{02}(0,\omega',t))^{-1} c(0,\omega',t) \neq 0$ is a constant because both $\overset{+}{L}_{02}$ and c are homogeneous of order $(0,0,r_3)$ and where $q^0_{kj}(\rho,\omega')$ are functions such that $\lim_{\rho\downarrow 0} q^0_{kj}(\rho,\omega') = q^0_{kj}(\rho,\omega')$ with $q^0_{kj}(\omega')$ defined by (3.2.14).

For $r_2 < j \le r_2 + r_3$, one has (3.2.25) with c replaced by 1. Hence,

$$(3.2.27)\qquad \pi_0 \text{ Op } L_{k0} 1_0 v_j = \langle\xi'\rangle^{\gamma_k+m_k+j-r_2} \delta_2 \quad q^0_{kj}(\rho,\omega')$$
$$+ \varepsilon^{-(\gamma_k+m_k+j-r_2)} \delta_3 \quad q_{kj}(\rho,\omega'),\ r_2 < j \le r_2+r_3.$$

Here $\delta_2 = (\overset{+}{L}_{02}(0,\omega',t))^{-1}$, $\delta_3 = (\overset{+}{L}_{01}(1,0,t))^{-1}\, t^{r_2}$ are nonzero constants, and $q^0_{kj}(\rho,\omega')$, $q_{kj}(\rho,\omega')$ are functions such that $\lim_{\rho\downarrow 0} q^0_{kj}(\rho,\omega') = q^0_{kj}(\omega')$, $\lim_{\rho\downarrow 0} q_{kj}(\rho,\omega') = q_{kj}$, with $q^0_{kj}(\omega')$, q_{kj} defined by (3.2.14), (3.2.15).

For $f \equiv 0$ and $\rho = \varepsilon\langle\xi'\rangle \le \rho_0$, we seek the solution of (3.2.2), (3.2.3) in the form

$$(3.2.28)\qquad u(\varepsilon,\widehat{\xi'},x_n) = \sum_{1\le j\le r_2+r_3} C_j(\varepsilon,\widehat{\xi'})\ v_j(\varepsilon,\widehat{\xi'},x_n),\ x_n > 0,$$

where the complex-valued functions C_j, $1 \le j \le r_2+r_3$, have to be determined.

With $N_1(\lambda) = \sum_{1\le j\le r_2} C_j\lambda^{j-1}$, $N_2(\lambda) = \sum_{r_2<j\le r_2+r_3} C_j\lambda^{j-1}$, this representation can be rewritten as follows:

$$u(\varepsilon,\widehat{\xi'},x_n) = \frac{1}{2\pi i}\int_\gamma \frac{e^{ix_n\lambda} N_1(\lambda)d\lambda}{\overset{+}{L}_{01}(\varepsilon,\widehat{\xi'},\lambda)d(\varepsilon,\widehat{\xi'},\lambda)} +$$
$$+ \frac{1}{2\pi i}\int_\gamma \frac{e^{ix_n\lambda} N_2(\lambda)d\lambda}{\overset{+}{L}_{01}(\varepsilon,\widehat{\xi'},\lambda)\overset{+}{L}_{02}(\varepsilon,\widehat{\xi'},\lambda)}$$

In the differential case (where, of course, $d \equiv 1$), the last formula is equivalent to the representation (3.2.8) in [Fr 1]:

$$u(\varepsilon,\widehat{\xi'},x_n) = \frac{1}{2\pi i}\int_\gamma \frac{e^{ix_n\lambda}\widetilde{N}_1(\lambda)d\lambda}{L_{01}^+(\varepsilon,\widehat{\xi'},\lambda)} + \frac{1}{2\pi i}\int_\gamma \frac{e^{ix_n\lambda}\widetilde{N}_2(\lambda)d\lambda}{L_{02}^+(\varepsilon,\widehat{\xi'},\lambda)}$$

where $N_1 = \widetilde{N}_1$ and $N_2(\lambda) = L_{01}^+(\varepsilon,\widehat{\xi'},\lambda)\widetilde{N}_2$, with $\widetilde{N}_2$ a polynomial of degree less than r_3.

As a consequence of (3.2.26), (3.2.27), one obtains the following system for the unknown functions C_j:

$$\sum_{1\le j\le r_2} (C_j\langle\xi'\rangle^{j-r_2})\,(\delta_1 q_{kj}^0(\rho,\omega') + O(\rho^{k_0-(m_k+j-r_2)})) +$$

$$+ \sum_{r_2<j\le r_2+r_3} (C_j\varepsilon^{-(j-r_2)})\,(\rho^{j-r_2}\delta_2 q_{kj}^0(\rho,\omega') + \rho^{-m_k}\delta_3 q_{kj}(\rho,\omega'))$$

$$= \varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\phi_k(\varepsilon,\xi'),\quad 1 \le k \le r_2.$$

$$\sum_{1\le j\le r_2} (C_j\langle\xi'\rangle^{j-r_2})(\rho^{m_k}\delta_1 q_{kj}^0(\rho,\omega') + O(\rho^{k_0+r_2-j})) +$$

$$+ \sum_{r_2<j\le r_2+r_3} (C_j\varepsilon^{-(j-r_2)})\,(\rho^{m_k+j-r_2}\delta_2 q_{kj}^0(\rho,\omega') + \delta_3 q_{kj}(\rho,\omega'))$$

$$= \varepsilon^{\gamma_k+m_k}\phi_k(\varepsilon,\xi')\ ,\quad r_2 < k \le r_2 + r_3.$$

With the notation $\psi_j = C_j\langle\xi'\rangle^{j-r_2}$, $1 \le j \le r_2$, $\psi_j = \varepsilon^{-(j-r_2)}C_j$, $r_2 < j \le r_2+r_3$, and with α,β given by (3.2.11), this system can be rewritten as follows:

$$T_{\rho,\omega'}\psi = \begin{pmatrix} A_{11}(\rho,\omega') & \rho^{-\alpha}\ A_{12}(\rho,\omega') \\ \rho^{\beta}A_{21}(\rho,\omega') & A_{22}(\rho,\omega') \end{pmatrix}\psi =$$

$$(3.2.29)\qquad = \begin{pmatrix} \operatorname{diag}(\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k})_1^{r_2} \\ \operatorname{diag}(\varepsilon^{\gamma_k+m_k})_{r_2+1}^{r_2+r_3} \end{pmatrix}\phi$$

where the matrices $A_{kj}(\rho,\omega') \in \mathrm{Hom}(\mathbb{C}^{r_{j+1}}, \mathbb{C}^{r_{k+1}})$, $1 \le k, j \le 2$, depend continuously upon the parameters $\rho \in [0,\rho_1]$, $\omega' \in \Omega_{n-1}$. As a consequence of the conditions (i), (ii), the matrices $A_{11}(0,\omega')$, $A_{22}(0,\omega')$ are invertible. Therefore, there exists $\rho_1' > 0$ such that for $\forall\, \rho \in [0,\rho_1']$, $\forall\, \omega' \in \Omega_{n-1}$, the inverses of the matrices $A_{11}(\rho,\omega')$, $A_{22}(\rho,\omega')$ exist and are uniformly bounded.

With

$$S_{\rho,\omega'} = \begin{pmatrix} A_{11}(\rho,\omega') & 0 \\ 0 & A_{22}(\rho,\omega') \end{pmatrix},$$

the matrix $T_{\rho,\omega'} S^{-1}_{\rho,\omega'}$ has the form

$$T_{\rho,\omega'}\, S^{-1}_{\rho'\omega'} = \begin{pmatrix} \mathrm{Id}_{r_2} & \rho^{-\alpha}\, \tilde{A}_{12}(\rho,\omega') \\ \rho_\beta\, \tilde{A}_{21}(\rho,\omega') \quad , & \mathrm{Id}_{r_3} \end{pmatrix}$$

where Id_s is the identity matrix in $\mathrm{Hom}\,(\mathbb{C}^s, \mathbb{C}^s)$ and $\tilde{A}_{kj} \in \mathrm{Hom}(\mathbb{C}^{r_{j+1}}, \mathbb{C}^{r_{k+1}})$ are continuous functions of the parameters ρ,ω'. The result established in [Fr 1] and formulated here as Lemma 3.1.32 implies that $(T_{\rho,\omega'}\, S^{-1}_{\rho,\omega'})^{-1}$ exists for $\rho \in (0,\rho_0]$, $\omega' \in \Omega_{n-1}$ with ρ_0 sufficiently small, and that

$$(T_{\rho,\omega'}\, S^{-1}_{\rho,\omega'})^{-1} = \mathrm{Id}_{r_2+r_3} + \rho^{\beta-\alpha} \begin{pmatrix} B_{11}(\rho,\omega') & 0 \\ 0 & B_{22}(\rho,\omega') \end{pmatrix} + \begin{pmatrix} 0 & \rho^{-\alpha} \quad B_{12}(\rho,\omega') \\ \rho^{\beta} \quad B_{21}(\rho,\omega') & 0 \end{pmatrix}$$

with continuous functions $B_{kj}(\rho,\omega') \in \mathrm{Hom}(\mathbb{C}^{r_{j+1}}, \mathbb{C}^{r_{k+1}})$. Hence, $|C_j|$ can be estimated as follows:

$$
(3.2.30)\begin{cases} |C_j(\varepsilon,\hat{\xi}')| \le C\langle\xi'\rangle^{r_2-j}\Big(\sum\limits_{1\le k\le r_2} \varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}|\phi_k| + \sum\limits_{r_2<k\le r_2+r_3} \varepsilon^{\gamma_k+m_k}\rho^{-\alpha}|\phi_k|\Big), 1\le j\le r_2 \\ |C_j(\varepsilon,\hat{\xi}')| \le C\varepsilon^{j-r_2}\Big(\sum\limits_{1\le k\le r_2} \varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\rho^{\beta}|\phi_k| + \sum\limits_{r_2<k\le r_2+r_3} \varepsilon^{\gamma_k+m_k}|\phi_k|\Big), r_2<j\le r_2+r_3. \end{cases}
$$

Thus, we have proved

Lemma 3.2.4. Let $f \equiv 0$, $\varepsilon\langle\xi'\rangle \le \rho_0$ with $\rho_0 > 0$ the constant fixed above. Then the solution of (3.2.2), (3.2.3) is well defined and has the form (3.2.28), where the functions v_j are given by (3.2.24) and where C_j satisfy (3.2.30).

We are now going to prove (3.2.23) in the case $f \equiv 0$ and $\rho = \varepsilon\langle\xi'\rangle \le \rho_0$, where $\rho_0 \in (0,\rho_1]$ is the constant in Lemma 3.1.32.

Following again [Fr 1], we rewrite the functions v_j, $1 \le j \le r_2$, as

$$
v_j(\varepsilon,\hat{\xi}',x_n) = (2\pi i)^{-1}\langle\xi'\rangle^{j-r_2} \int_{\Gamma_1} \frac{e^{ix_n\langle\xi'\rangle t}\, t^{j-1}dt}{L^+_{01}(\rho,\omega',t)d(\rho,\omega',t)}
$$

$$
+ (2\pi i)^{-1}\varepsilon^{-(j-r_2)} \int_{\Gamma_2} \frac{e^{ix_n\varepsilon^{-1}t}\, t^{j-1}r(1,\rho\omega',t)}{L^+_{01}(1,\rho\omega',t)d(1,\rho\omega',t)}\, dt,
$$

where $r = 1\text{-}d$ satisfies (3.1.22). Hence,

$$
\begin{aligned}
||v_j(\varepsilon,\hat{\xi}',x_n)||^+_{(s),\xi'} &\le C\,\varepsilon^{-s_1}(\langle\xi'\rangle^{j-r_2+s_2-\frac12} + \varepsilon^{-(j-r_2+s_2-\frac12)}(\varepsilon\langle\xi'\rangle)^{k_0}) \\
&\le C\,\varepsilon^{-s_2}\langle\xi'\rangle^{j-r_2+s_2-\frac12}(1+\rho^{k_0+\frac12+r_2-j-s_2}) \\
&\le C\,\varepsilon^{-s_1}\langle\xi'\rangle^{j-r_2+s_2-\frac12}
\end{aligned}
$$

since $s_2 < k_0+\frac12$. Therefore (3.2.30) implies that for $1 \le j \le r_2$, one has:

$$||C_jv_j||^+_{(s),\xi'} \le C\varepsilon^{\gamma_k - s_1}\Big(\sum_{1\le k\le r_2} \langle\xi'\rangle^{s_2-m_k-\frac12}|\phi_k| + \sum_{r_2<k\le r_2+r_3} \varepsilon^{m_k}\rho^{-m_{r_2}}\langle\xi'\rangle^{s_2-\frac12}|\phi_k|\Big)$$

$$\le C\varepsilon^{\gamma_k - s_1}\Big(\sum_{1\le k\le r_2} \langle\xi'\rangle^{s_2-m_k-\frac12}|\phi_k| + \rho^{s_2-m_{r_2}-\frac12}\sum_{r_2<k\le r_2+r_3} \varepsilon^{m_k-s_2-\frac12}|\phi_k|\Big)$$

$$\le C\Big(\sum_{1\le k\le r_2}[\phi_k]_{(\tau_k),\xi'} + \sum_{r_2<k\le r_2+r_3}[\phi_k]_{(\sigma_k),\xi'}\Big)$$

since $m_{r_2} + \frac12 < s_2$ and since $\varepsilon\langle\xi'\rangle \le \rho_0$ implies that $C^{-1} \le \langle\varepsilon\xi'\rangle \le C$.

Following again [Fr 1], we rewrite the functions $v_j, r_2 < j \le r_2+r_3$, as follows:

$$v_j(\varepsilon,\hat{\xi}',x_n) = (2\pi i)^{-1}\langle\xi'\rangle^{j-r_2}\int_{\Gamma_1}\frac{e^{ix_n\langle\xi'\rangle t}\,t^{j-1}dt}{L^+_{01}(\rho,\omega',t)\,L^+_{02}(\rho,\omega',t)} +$$

$$+\ (2\pi i)^{-1}\,\varepsilon^{-(j-r_2)}\int_{\Gamma_2}\frac{e^{ix_n\varepsilon^{-t}t}\,t^{j-1}dt}{L^+_{01}(1,\rho\omega',t)L^+_{02}(1,\rho\omega',t)}\ .$$

Hence,

$$||v_j(\varepsilon,\hat{\xi}',x_n)||^+_{(s),\xi'} \le C\,\varepsilon^{-s_1}(\langle\xi'\rangle^{j-r_2+s_2-\frac12} + \varepsilon^{-(j-r_2+s_2-\frac12)})$$

$$\le C\,\varepsilon^{-(s_1+s_2+j-r_2-\frac12)}(1+\rho^{j-r_2+s_2-\frac12})$$

$$\le C\,\varepsilon^{-(s_1+s_2+j-r_2-\frac12)}$$

since $s_2 > -\frac12$. Therefore (3.2.30) implies that for $r_2 < j \le r_2+r_3$, one has:

$$||C_jv_j||^+_{(s),\xi'} \le C\,\varepsilon^{-(s_1+s_2-\frac12)}\Big(\sum_{1\le k\le r_2}\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\rho^{\beta}|\phi_k| + \sum_{r_2<k\le r_2+r_3}\varepsilon^{\gamma_k+m_k}|\phi_k|\Big)$$

$$\le C\Big(\rho^{\beta-s_2+\frac12}\sum_{1\le k\le r_2}\varepsilon^{\gamma_k-s_1}\langle\xi'\rangle^{s_2-m_k-\frac12}|\phi_k| + \sum_{r_2<k\le r_2+r_3}\varepsilon^{-(s_1+s_2-\frac12-\gamma_k-m_k)}|\phi_k|\Big)$$

$$\le C\Big(\sum_{1\le k\le r_2}[\phi_k]_{(\tau_k),\xi'} + \sum_{r_2<k\le r_2+r_3}[\phi_k]_{(\sigma_k),\xi'}\Big)$$

since $s_2 < \beta + \frac{1}{2}$ and since $\varepsilon\langle\xi'\rangle \le \rho_0$ implies that $C^{-1} \le \langle\varepsilon\xi'\rangle \le C$.
Thus, the following statement is proved:

Lemma 3.2.5. Let $f \equiv 0$, $\varepsilon \langle\xi'\rangle \le \rho_0$ with $\rho_0 > 0$ the constant fixed above. Then the solution of (3.2.2), (3.2.3) satisfies the second of the inequalities (3.2.23).

Let now $\rho = \varepsilon \langle\xi'\rangle \ge \rho_0 > 0$, $f \equiv 0$. In this case, the proof of the a-priori-estimate for differential boundary value problems in [Fr 1] can be extended immediately. The general solution of (3.2.2), (3.2.3) with $f \equiv 0$ which decreases at infinity can be written as follows:

$$(3.2.31) \qquad u(\varepsilon,\hat{\xi}',x_n) = \frac{1}{2\pi i} \int_{\gamma(\varepsilon,\xi')} \frac{e^{ix_n\lambda} M_1(\varepsilon,\hat{\xi}',\lambda)\, d\lambda}{L_0^+(\varepsilon,\hat{\xi}',\lambda)}$$

where $\lambda \to M_1(\varepsilon,\hat{\xi}',\lambda)$ is a polynomial of degree less than r_2+r_3 and where γ is a Jordan contour which encloses all the zeroes of $L_0^+(\varepsilon,\hat{\xi}',\lambda)$.

Following [Fr 1], we are going to rewrite the representation (3.2.31). Let the curve $\Gamma = \Gamma(\rho_0)$ be defined as in Lemma 3.1.14(iii). Introducing the variable $t = \langle\xi'\rangle^{-1}\lambda$ and using the homogenity of L_0^+, one finds that

$$(3.2.32) \qquad u(\varepsilon,\hat{\xi}',x_n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{ix_n\langle\xi'\rangle t} M(\varepsilon,\rho,\omega',t)}{\rho^{-r_3} L_0^+(\rho,\omega',t)}$$

where $t \to M(\varepsilon,\rho,\omega',t)$ is a polynomial of degree less than r_2+r_3.

For $1 \le k \le r_2+r_3$, we are going to construct polynomials

$$M_k(\rho,\omega',t) = \sum_{1\le l\le r_2+r_3} C_{kl}(\rho,\omega')\, t^{l-1}$$

which satisfy the orthogonality conditions

$$(3.2.33)\qquad \frac{1}{2\pi i}\int_\Gamma \frac{\rho^{\gamma_j-p_j}L_{j0}(\rho,\omega',t)\;M_k(\rho,\omega',t)dt}{\rho^{-r_3}\;L_0^+(\rho,\omega',t)} = \delta_{jk},$$

$$1 \le j,\ k \le r_2+r_3,\ \forall\ (\rho,\omega') \in [\rho_0,\infty) \times \Omega_{n-1}$$

where $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ otherwise.

With the matrix $||Q_{kj}(\rho,\omega')||_{1\le k,\ j\le r_2+r_3}$ introduced in (3.2.19), one has:

$$\frac{1}{2\pi i}\int_\Gamma \frac{\rho^{\gamma_3-p_j}L_{j0}(\rho,\omega',t)\;M_k(\rho,\omega',t)dt}{\rho^{-r_3}\;L_0^+(\rho,\omega',t)} = \sum_{1\le l\le r_2+r_3} C_{kl}(\rho,\omega')\;Q_{lj}(\rho,\omega'),$$

$$1 \le j,\ k \le r_2+r_3.$$

We choose $C(\rho,\omega') = ||C_{kl}(\rho,\omega')||_{1\le k,l\le r_2+r_3}$ to be the inverse matrix of $||Q_{kj}(\rho,\omega')||_{1\le k,j\le r_2+r_3}$ for $\rho \in [\rho_0,\infty)$, $\omega' \in \Omega_{n-1}$.

Since the condition (iii) implies that the matrix

$$\lim_{\rho\to\infty} ||Q_{kj}(\rho,\omega')||_{1\le k,j\le r_2+r_3} = ||q_{kj}^{00}(\omega')||_{1\le k,j\le r_2+r_3}$$

is nonsingular for $\forall\ \omega' \in \Omega_{n-1}$, the functions $C_{kl}(\rho,\omega')$, $1 \le k,\ l \le r_2+r_3$, are uniformly bounded with respect to $\rho \in [\rho_0,\infty)$, $\omega' \in \Omega_{n-1}$.

Let the functions v_k, $1 \le k \le r_2+r_3$, be given by

$$v_k(\varepsilon,\hat{\xi}',x_n)=\frac{1}{2\pi i}\int_\Gamma \frac{e^{ix_n<\xi'>t}\;M_k(\rho,\omega',t)\;dt}{\rho^{-r_3}\;L_0^+(\rho,\omega',t)}\;,\; x_n > 0.$$

The homogenity of L_{j0} and (3.2.33) yield:

$$\pi_0\;L_{j0}(\varepsilon,\hat{\xi}',-i\frac{\partial}{\partial x_n})l_0\;v_k = \frac{1}{2\pi i}\int_\Gamma \frac{L_{j0}(\varepsilon,\hat{\xi}',<\xi'>t)\;M_k(\rho,\omega',t)\;dt}{\rho^{-r_3}\;L_0^+(\rho,\omega',t)}$$

$$= \frac{<\xi'>^{\gamma_j+m_j}}{2\pi i}\int_\Gamma \frac{L_{j0}(\rho,\omega',t)\;M_k(\rho,\omega',t)\;dt}{\rho^{-r_3}\;L_0^+(\rho,\omega',t)} = <\xi'>^{\gamma_j+m_j}\rho^{p_j-\gamma_j}\;\delta_{kj}.$$

Thus, we have proved

Lemma 3.2.6. Let $f \equiv 0$, $\varepsilon <\xi'> \geq \rho_0$ with ρ_0 the constant fixed above. Then the solution of (3.2.2), (3.2.3) is well defined and can be written as follows:

$$u(\varepsilon,\widehat{\xi'},x_n) = \sum_{1\leq k\leq r_2+r_3} v_k(\varepsilon,\widehat{\xi'},x_n)\varepsilon^{\gamma_k}<\xi'>^{-m_k}\rho^{-p_k}\phi_k(\varepsilon,\widehat{\xi'}),\ x_n > 0.$$

Since $\lim_{\rho\to\infty} \rho^{-r_3} L_0^+(\rho,\omega',t) = L_{00}^+(\omega',t)$, the function $\Gamma \ni t \to |\rho^{-r_3} L_0^+(\rho,\omega',t)|$ is bounded from below by a positive constant which does not depend upon $\rho \in [\rho_0,\infty]$, $\omega' \in \Omega_{n-1}$. Since for $1 \leq k \leq r_2+r_3$, the function $\Gamma \ni t \to |M_k(\rho,\omega',t))|$ is bounded from above uniformly with respect to $\rho \in [\rho_0,\infty]$, $\omega' \in \Omega_n$, $||v_k||^+_{(s)\xi'}$ can be estimated as follows:

$$||v_k(\varepsilon,\widehat{\xi'},x_n)||^+_{(s),\xi'} \leq C\,\varepsilon^{-s_1}<\xi'>^{s_2-\frac{1}{2}}<\varepsilon\xi'>^{s_3}\ \forall(\varepsilon,\xi') \in (0,\varepsilon_0]\times\mathbb{R}^{n-1}\setminus\{0\},$$
$$\varepsilon<\xi'> \geq \rho_0.$$

Hence,

$$||u(\varepsilon,\widehat{\xi'},x_n)||^+_{(s)\xi'} \leq C \sum_{1\leq k\leq r_2+r_3} \varepsilon^{\gamma_k-s_1}<\xi'>^{s_2-m_k\pm\frac{1}{2}}<\varepsilon\xi'>^{s_3}\rho^{-p_k}|\phi_k(\varepsilon,\widehat{\xi'})|$$
$$\forall(\varepsilon,\xi')\in(0,\varepsilon_0]\times\mathbb{R}^{n-1}\setminus\{0\},\ \varepsilon<\xi'> \geq \rho_0.$$

Since for $\rho = \varepsilon<\xi'> \geq \rho_0 > 0$ and $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 \leq 1$ there is the equivalency

$$\rho^2 = \varepsilon^2<\xi'>^2 \leq <\varepsilon\xi'>^2 \leq (1+\rho_0^{-2})\ \varepsilon^2<\xi'>^2 = (1+\rho_0^{-2})\rho^2,$$

one has:

$$||u(\varepsilon,\widehat{\xi'},x_n)||^+_{(s),\xi'} \leq C \sum_{1\leq k\leq r_2+r_3} \varepsilon^{\gamma_k-s_1}<\xi'>^{s_2-m_k-\frac{1}{2}}<\varepsilon\xi'>^{s_3-p_k}|\phi_k(\varepsilon,\widehat{\xi'})|$$
$$\leq C\Big(\sum_{1\leq k\leq r_2} [\phi_k]_{(\tau_k),\xi'} + \sum_{r_2<k\leq r_2+r_3} [\phi_k]_{(\sigma_k),\xi'}\Big).$$

Thus, we have proved

Lemma 3.2.7. For $f \equiv 0$, $\varepsilon \langle\xi'\rangle \geq \rho_0$ with $\rho_0 > 0$ chosen above, the solution of (3.2.2), (3.2.3) satisfies the second of the inequalities (3.2.23).

Now we drop the assumption $f \equiv 0$. For given $f \in H_{(s-\nu),\xi'}(\mathbb{R}_+)$, we are going to construct a solution $w \in H_{(s),\xi'}(\mathbb{R}_+)$ of the equation (3.2.2).

Let $\lambda \to c(\varepsilon,\widehat{\xi'},\lambda)$ be the polynomial constructed in the proof of Lemma 3.1.21. Then one has $d = c^{-1} L_{02}^{+} \in D^{k_0}_{(0,0,0)}$ and (3.2.2) can be rewritten as follows:

$$\pi_+(L_0^- c)(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 \pi_+(L_{01}^+ d)(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) u_+(x_n) = f(x_n),\ x_n>0.$$

Indeed, since $\lambda \to (L_0^- c)(\varepsilon,\widehat{\xi'},\lambda)$ is analytic for $\operatorname{Im}\lambda > 0$, one has $\pi_+(L_0^- c)(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})v(x_n) \equiv 0\ \forall x_n > 0$ if $\operatorname{supp} v \subset \mathbb{R}_-$. Now let $l : H_{(s-\nu),\xi'}(\mathbb{R}_+) \to H_{(s-\nu),\xi'}(\mathbb{R})$ be a bounded extension operator.

Lemma 3.2.8. The function w defined by

$$w(x_n) = \pi_+((L_{01}^+ d)^{-1})(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 \pi_+((L_0^- c)^{-1})(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) lf \tag{3.2.34}$$

satisfies (3.2.2), and, moreover, $w \in H_{(s),\xi'}(\mathbb{R}_+)$.

Proof. Since the function $\lambda \to L_1(\varepsilon,\widehat{\xi'},\lambda) = (L_{01}^+ d)(\varepsilon,\widehat{\xi'},\lambda)$ has no zeroes for $\operatorname{Im}\lambda < 0$, one has $\operatorname{supp}\,(L_1)^{-1}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})\ l_0 f \subset \mathbb{R}_+$ for any function f defined in $\mathbb{R}_+$. Hence,

$$\pi_+ L_1(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 \pi_+ (L_1)^{-1}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 f = f.$$

Moreover, since $\lambda \to L_2(\varepsilon,\widehat{\xi'},\lambda) = (L_0^- c)(\varepsilon,\widehat{\xi'},\lambda)$ is analytic for $\operatorname{Im}\lambda > 0$, one has $\pi_+ L_2(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})v(x_n) = 0\ \ \forall x_n > 0$ if $\operatorname{supp} v \subset \mathbb{R}_-$. Hence,

$$\pi_+ L_2(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0\pi_+(L_2)^{-1}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1\ f = f$$

for any function f defined in $\mathbb{R}_+$. Therefore, w satisfies the equation (3.2.2). Now we are going to prove that $w \in H_{(s),\xi'}(\mathbb{R}_+)$. The ellipticity of $L_0^- c$ of order (ν_1,ν_2-r_2,ν_3) implies that

$$(3.2.35)\qquad ||\pi_+(L_0^-c)^{-1}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1f||^+_{(s-r_2e_2),\xi'} \le C||f||^+_{(s-\nu),\xi'}$$

$$\forall f \in H_{(s-\nu),\xi'},\ \forall(\varepsilon,\xi')\in(0,\varepsilon_0]\times\mathbb{R}^{n-1}\setminus\{0\}.$$

The inclusion $d(\varepsilon,\xi) \in \mathcal{D}^{k_0}_{(0,0,0)}$ and Lemma 3.1.21 imply that d^{-1} is in the class $\mathcal{D}^{k_0}_{(0,0,0)}$, too. As a consequence of Lemma 3.1.18, $(L_{01}^+d)^{-1} \in \mathcal{D}^{k_0}_{(0,-r_2,0)}$. Therefore, with $v = \pi_+(L_0^-c)(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1f \in H_{(s-r_2e_2),\xi'}(\mathbb{R}_+)$, one has:

$$(3.2.36)\qquad ||\pi_+((L_{01}^+d)^{-1})(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 v||^+_{(s),\xi'} \le C||v||^+_{(s-r_2e_2),\xi'}$$

$$\forall(\varepsilon,\xi')\in(0,\varepsilon_0]\times\mathbb{R}^{n-1}\setminus\{0\}.$$

The inequalities (3.2.35),(3.2.36) imply that

$$(3.2.37)\qquad ||w||^+_{(s),\xi'} \le C||\pi_+(L_0^-c)^{-1}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1f||^+_{(s-r_2e_2),\xi'}$$

$$\le C||f||^+_{(s-\nu),\xi'}\ \ \forall f\in H_{(s-\nu),\xi'},\forall(\varepsilon,\xi')\in(0,\varepsilon_0]\times\mathbb{R}^{n-1}\setminus\{0\}.$$

□

Proof of Theorem 3.2.1.

The solution of (3.2.2), (3.2.3) has the following form:

$$(3.2.38)\ u(\varepsilon,\widehat{\xi'},x_n) = w(\varepsilon,\widehat{\xi'},x_n) + v(\varepsilon,\widehat{\xi'},x_n)$$

where w is given by (3.2.34) and where v is a solution of the problem

$$\begin{aligned}&\pi_+L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0\ v(x_n) = 0 &&,\ x_n > 0\\ &\pi_0L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0\ v = \widetilde{\phi}_j(\varepsilon,\widehat{\xi'}) &&,\ 1 \le j \le r_2+r_3\end{aligned}$$

with

$$\widetilde{\phi_j}(\varepsilon,\widehat{\xi'}) = \phi_j(\varepsilon,\widehat{\xi'}) - \pi_0 \, L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 w, \quad 1 \le j \le r_2+r_3.$$

The trace theorem, which was established in [Fr 1] and formulated in section 2 of this paper, implies that for $1 \le j \le l_0$:

$$\begin{aligned} [\widetilde{\phi_j}]_{(\tau_j)\xi'} &\le [\phi_j]_{(\tau_j),\xi'} + [\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) l_0 w]_{(\tau_j),\xi'} \\ &\le [\phi_j]_{(\tau_j),\xi'} + C||w||^+_{(s),\xi'} \\ &\le [\phi_j]_{(\tau_j),\xi'} + C||f||^+_{(s-\nu),\xi'} \, . \end{aligned}$$

Similarly, one has for $l_0 < j \le r_2+r_3$:

$$[\widetilde{\phi_j}]_{(\sigma_j),\xi'} \le [\phi_j]_{(\sigma_j),\xi'} + C||f||^+_{(s-\nu),\xi'} \, .$$

Since (3.2.23) has already been proved for $f \equiv 0$, one gets, using also (3.2.37):

$$\begin{aligned} ||u||^+_{(s),\xi'} &\le ||w||^+_{(s),\xi'} + ||v||^+_{(s),\xi'} \\ &\le C(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le l_0} [\widetilde{\phi_j}]_{(\tau_j),\xi'} + \sum_{l_0<j\le r_2+r_3} [\widetilde{\phi_j}]_{(\sigma_j),\xi'}) \\ &\le C(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le l_0} [\phi_j]_{(\tau_j),\xi'} + \sum_{l_0<j\le r_2+r_3} [\phi_j]_{(\sigma_j),\xi'}). \end{aligned}$$

We now turn to the case when the condition $s_2 > r_2-\frac{1}{2}$ in (3.2.21) is violated.

Consider the following boundary value problem:

$$(3.2.39) \qquad \pi_+ L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+(x_n) = f(x_n),\ x_n > 0$$

$$(3.2.40) \qquad \pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+ = \phi_j,\ 1 \le j \le l_1.$$

Here $l_1 \ge 0$ is not necessarily equal to r_2+r_3, L_0, L_{j0} are as above and the solution u_+ is sought in the space $\overset{\circ}{H}{}^+_{(s),\xi'}$ with

$$(3.2.41) \qquad r_2+r_3-l_1-\frac{1}{2} < s_2+s_3 < r_2+r_3-l_1+\frac{1}{2}$$

(see [V-E] where problems of this type without small parameter have been systematically treated). In order to have the traces in (3.2.40) well defined, it is assumed that

$$(3.2.42) \qquad \max_{1\le j\le l_1} (m_j+p_j) + \frac{1}{2} < s_2+s_3.$$

Let $l, 0 \le l \le l_1$, be such that with $m_0 = -\infty$, $m_{l_1+1} = +\infty$, one has

$$(3.2.43) \qquad m_l \le r_2-l-1 < m_{l+1}.$$

The number l is unique, if it exists, since $(m_l)_l$ and $(r_2-l-1)_l$ are monotonically increasing and strictly monotonically decreasing, respectively. In this case, we introduce l_0,α,β by

$$(3.2.44) \qquad \begin{cases} l_0 \overset{\text{def}}{=} l \\ \alpha \overset{\text{def}}{=} r_2-l_0-1 \\ \beta \overset{\text{def}}{=} r_2-l_0 \end{cases}$$

The reduced problem of (3.2.39), (3.2.40) is given as follows:

$$\text{(3.2.45)} \qquad \pi_+ \, L_0^0(\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+(x_n) = f(x_n), \quad x_n > 0$$

$$\text{(3.2.46)} \qquad \pi_0 \, L_0^0(\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+ = \phi_j, \quad 1 \le j \le l_0$$

Here the solution u_+ is sought in the space $\overset{\circ}{H}_{s_2,\xi'}$ with

$$\text{(3.2.47)} \qquad \alpha + \tfrac{1}{2} < s_2 < \beta + \tfrac{1}{2}.$$

Note that the traces in (3.2.46) are well defined as a consequence of the first inequality in (3.2.43).

Theorem 3.2.9. Assume that the function $L_0(\varepsilon,\xi')$ is an elliptic singularly perturbed rational symbol of order ν, that (3.2.43) holds for some $l = l_0 \in \{0,\dots l_1\}$ and that the boundary symbols L_{j0} satisfy the coerciveness condition (i)-(iv). Then the problem (3.2.39), (3.2.40) has a unique solution $u_+ \in \overset{\circ}{H}{}^+_{(s)}$ for $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, which satisfies the following two-sided estimate:

$$\text{(3.2.48)} \qquad C^{-1}(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le l_0} [\phi_j]_{(\tau_j),\xi'} + \sum_{l_0<j\le l_1} [\phi_j]_{(\sigma_j),\xi'}) \le$$

$$\le ||u_+||_{(s)\xi'} \le C(||f||^+_{(s-\nu),\xi'} + \sum_{1\le j\le l_0} [\phi_j]_{(\tau_j),\xi'} +$$

$$+ \sum_{l_0<j\le l_1} [\phi_j]_{(\sigma_j),\xi'}).$$

Here $s = (s_1,s_2,s_3)$ satisfies (3.2.41), (3.2.47), τ_j,σ_j are as in Theorem 3.2.1 and C is a constant which does not depend upon $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$.

Remark 3.2.10. As the Theorem 3.2.1 above, this result is in its formulation and proof similar to Theorem 3.2.11 in [Fr 1]. A difference, however, lies in the fact, that the perturbed problem contains l_1 ($\neq r_2+r_3$, in general) boundary conditions (see [V-E]). Moreover, the number l_0 of boundary conditions in the reduced problem is, in general, not r_2, but is determined from (3.2.43). Note that in Theorem 3.2.9, there is no restriction of s_2 due to k_0 (the number up to which the symbol L_0 satisfies the extension of the smoothness condition in [V-E] to symbols with small parameter).

Proof of Theorem 3.2.9. Since L_0 has order ν, the operator $L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})$ is bounded from $\overset{\circ}{H}{}^{+}_{(s),\xi'}$ to $H_{(s-\nu),\xi'}(\mathbb{R})$. Hence,

$$||f||^{+}_{(s-\nu),\xi'} = ||\pi_+ L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+||^{+}_{(s),\xi'}$$

$$\leq ||L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+||_{(s),\xi'}$$

holds with a constant C independent upon ε,ξ',u_+. Moreover, the trace theorem established in [Fr 1] and stated here as Theorem 2.2.4 yields

$$[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+]_{(\tau_j),\xi'} \leq C||u_+||_{(s),\xi'}, \quad 1 \leq j \leq l_0$$

$$[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})u_+]_{(\sigma_j),\xi'} \leq C||u_+||_{(s),\xi'}, \quad l_0 < j \leq l_1.$$

This proves the first of the inequalities (3.2.48).

In order to prove the second part of (3.2.48), consider first the case that $f \equiv 0$, $\varepsilon<\xi'> \leq \rho_0$, where $\rho_0 > 0$ will be chosen later.

Introduce the functions v_j, $1 \leq j \leq l_1$, by

$$v_j(x_n) = \frac{1}{2\pi i} \int_\gamma \frac{e^{ix_n\lambda}\lambda^{j-1}}{L_0^{+}(\varepsilon,\widehat{\xi'},\lambda)} d\lambda$$

with γ as in (3.2.24). Using a technique introduced in [Fr 1], we obtain for $\rho \in (0,\rho_1]$, with ρ_1, Γ_1, Γ_2 being as in Lemma 3.1.14(ii):

$$\pi_0 L_{k0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})\, l_0 v_j = \frac{1}{2\pi i}\int_\gamma \frac{L_{k0}(\varepsilon,\widehat{\xi'},\lambda)\lambda^{j-1}d\lambda}{L_0^+(\varepsilon,\widehat{\xi'},\lambda)}$$

$$= \langle\xi'\rangle^{j-r_2+m_k+\gamma_k}\cdot\frac{1}{2\pi i}\cdot\int_{\Gamma_1}\frac{L_{k0}(\rho,\omega',t)t^{j-1}dt}{L_0^+(\rho,\omega',t)} +$$

$$+\ \varepsilon^{-(j-r_2+m_k+\gamma_k)}\cdot\frac{1}{2\pi i}\int_{\Gamma_2}\frac{L_{k0}(1,\rho\omega',t)t^{j-1}dt}{L_0^+(1,\rho\omega',t)}$$

$$= \varepsilon^{-\gamma_k}\langle\xi'\rangle^{j-r_2+m_k}\,\delta_1 q^0_{kj}(\rho,\omega') + \varepsilon^{-(j-r_2+m_k+\gamma_k)}\,\delta_2\, q_{kj}(\rho,\omega')$$

$$1 \le k,\ j \le l_1,$$

where $\delta_1 = (L^+_{02}(0,\omega',t))^{-1}$, $\delta_2 = t^{r_2}(L^+_{01}(1,0,t))^{-1}$ are nonzero constants and where $\lim_{\rho\downarrow 0} q^0_{kj}(\rho,\omega') = q^0_{kj}(\omega')$, $\lim_{\rho\downarrow 0} q_{kj}(\rho,\omega') = q_{kj}$ with $q^0_{kj}(\omega')$, q_{kj} defined in the coerciveness condition. Seeking the solution in the form

$$u_+(\varepsilon,\widehat{\xi'},x_n) = \sum_{1\le j\le l_1} C_j(\varepsilon,\widehat{\xi'})\ v_j(\varepsilon,\widehat{\xi'},x_n),$$

one obtains the following system for the unknown functions C_j, $1 \le j \le l_1$:

$$\sum_{1\le j\le l_0}(\delta_1 q^0_{kj} + \rho^{-(j-r_2+m_k)}\delta_2 q_{kj})(\langle\xi'\rangle^{j-r_2} C_j) +$$

$$+ \sum_{l_0<j\le l_1}(\rho^{j-r_2}\delta_1 q^0_{kj} + \rho^{-m_k}\delta_2 q_{kj})(\varepsilon^{-(j-r_2)} C_j) =$$

$$= \varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\phi_k, \quad 1 \le k \le l_0$$

$$\sum_{1\le j\le l_0}(\rho^{m_k}\delta_1 q^0_{kj} + \rho^{-(j-r_2)}\delta_2 q_{kj})(\langle\xi'\rangle^{j-r_2} C_j) +$$

$$+ \sum_{l_0<j\le l_1}(\rho^{j-r_2+m_k}\delta_1 q^0_{kj} + \delta_2 q_{kj})(\varepsilon^{-(j-r_2)} C_j) =$$

$$= \varepsilon^{\gamma_k+m_k}\phi_k, \quad l_0 < k \le l_1.$$

For $1 \le j$, $k \le l_0$, one has $j-r_2+m_k \le l_0-r_2-m_{l_0} < 0$ as a consequence of

(3.2.43). Moreover, for $l_0 < k, j \le l_1$, (3.2.43) implies that $j-r_2+m_k \ge l_0+1-r_2+m_{l_0+1} > 0$. For $1 \le k \le l_0$, $l_0 < j \le l_1$, one obtains $j-r_2 \ge l_0+1-r_2 = -\alpha > 0$, $-m_k \ge -m_{l_0} \ge l_0 + 1-r_2 = -\alpha$ and for $l_0 < k \le l_1$, $1 \le j \le l_0$, one has $m_k \ge m_{l_0+1} \ge r_2-l_0 = \beta$, $-(j-r_2) \ge -(l_0-r_2) = \beta$. Therefore, with $\psi_j = C_j\langle\xi'\rangle^{j-r_2}$, $1 \le j \le l_0$, $\psi_j = C_j\,\varepsilon^{-(j-r_2)}$, $l_0 < j \le l_1$, and with α,β given by (3.2.44), this system can be rewritten as follows:

$$\begin{pmatrix} A_{11}(\rho,\omega') & \rho^{-\alpha}A_{12}(\rho,\omega') \\ \rho^{\beta}A_{21}(\rho,\omega') & A_{22}(\rho,\omega') \end{pmatrix}\psi = \begin{pmatrix} \mathrm{diag}(\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k})_1^{l_0} \\ \mathrm{diag}(\varepsilon^{\gamma_k+m_k})_{l_0+1}^{l_1} \end{pmatrix}\phi$$

where the matrices $A_{11} \in \mathrm{Hom}(\mathbb{C}^{l_0},\mathbb{C}^{l_0})$, $A_{12} \in \mathrm{Hom}(\mathbb{C}^{l_1-l_0},\mathbb{C}^{l_0})$, $A_{21} \in \mathrm{Hom}(\mathbb{C}^{l_0},\mathbb{C}^{l_1-l_0})$ and $A_{22} \in \mathrm{Hom}(\mathbb{C}^{l_1-l_0},\mathbb{C}^{l_1-l_0})$ depend continuously upon the parameters ρ,ω'.

As a consequence of the conditions (i), (ii) and of the result established in [Fr 1] and stated here as Lemma 3.1.32, the solution of this system exists for $\rho \in (0,\rho_0]$ with $\rho_0 > 0$ sufficiently small and satisfies the estimate

$$(3.2.49)\quad \begin{cases} |C_j(\varepsilon,\hat{\xi}')| \le C\langle\xi'\rangle^{r_2-j}\Big(\sum\limits_{1\le k\le l_0}\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}|\phi_k| + \sum\limits_{l_0<k\le l_1}\varepsilon^{\gamma_k+m_k}\rho^{-\alpha}|\phi_k| & 1 \le j \le l_0 \\ |C_j(\varepsilon,\hat{\xi}')| \le C\,\varepsilon^{-(r_2-j)}\Big(\sum\limits_{1\le k\le l_0}\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\rho^{\beta}|\phi_k| + \sum\limits_{l_0<k\le l_1}\varepsilon^{\gamma_k+m_k}|\phi_k|\Big), & l_0 < j \le l_1 \end{cases}$$

Following [Fr 1], we rewrite the functions v_j, $1 \le j \le l_1$, as the sum of the integrals over the contours Γ_1,Γ_2 introduced above:

$$v_j(x_n) = \frac{<\xi'>^{j-r_2}}{2\pi i} \int_{\Gamma_1} \frac{e^{ix_n<\xi'>t}\, t^{j-1}dt}{L_0^+(\rho,\omega',t)} + \frac{\varepsilon^{-(j-r_2)}}{2\pi i} \int \frac{e^{ix_n\varepsilon^{-1}t}\, t^{j-1}dt}{L_0^+(1,\rho\omega',t)} .$$

Since for $1 \le j \le l_0$ one has $s_2+j-r_2-\frac{1}{2} \le s_2-\beta-\frac{1}{2} < 0$ and for $l_0 < j \le l_1$, $s_2+j-r_2-\frac{1}{2} \ge s_2-\alpha-\frac{1}{2} > 0$, the norm of v_j can be estimated as follows:

$$||v_j||_{(s),\xi'} \le C(<\xi'>^{s_2+j-r_2-\frac{1}{2}} + \varepsilon^{-(s_2+j-r_2-\frac{1}{2})})$$

$$\le \begin{cases} C<\xi'>^{s_2+j-r_2-\frac{1}{2}} & \text{if } 1 \le j \le l_0 \\ C\,\varepsilon^{-(s_2+j-r_2-\frac{1}{2})} & \text{if } l_0 < j \le l_1 \end{cases}$$

The estimate (3.2.49) yields that for $1 \le j \le l_0$,

$$||c_jv_j||_{(s),\xi'} \le C(\sum_{1\le k\le l_0} \varepsilon^{\gamma_k}<\xi'>^{s_2-m_k-\frac{1}{2}}|\phi_k|+\rho^{s_2-\alpha-\frac{1}{2}} \sum_{l_0<k\le l_1} \varepsilon^{-s_2+\gamma_k+m_k+\frac{1}{2}}|\phi_k|)$$

and for $l_0 < j \le l_1$,

$$||c_jv_j||_{(s),\xi'} \le C(\rho^{\beta+\frac{1}{2}-s_2} \sum_{1\le k\le l_0} \varepsilon^{\gamma_k}<\xi'>^{s_2-\frac{1}{2}-m_k}|\phi_k|+ \sum_{l_0<k\le l_1} \varepsilon^{-s_2+\gamma_k+m_k+\frac{1}{2}}|\phi_k|).$$

Thus, the second inequality in (3.2.48) is proved in the case $f \equiv 0$, $\rho \in (0,\rho_0]$, where $\rho_0 > 0$ has been fixed above. The case $f \equiv 0$, $\rho \ge \rho_0$ can be treated as in [Fr 1] (see also the proof of Lemma 3.2.7).

Following [V-E], we are going to construct a solution $w_+ \in \overset{\circ}{H}{}^+_{(s),\xi'}$ to the inhomogeneous equation (3.2.39).

Let $l : H_{(s-\nu),\xi'}(\mathbb{R}_+) \to H_{(s-\nu),\xi'}(\mathbb{R})$ be an extension operator which satisfies the inequality

$$||lf||_{(s-\nu),\xi'} \le C||f||^+_{(s-\nu),\xi'}\ \forall\ f \in H_{(s-\nu),\xi'}(\mathbb{R}_+),\ \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi' \in \mathbb{R}^{n-1}$$

with a constant C. With L_0^+, L_0^- as in (3.2.4), the function

$$w_+(\widehat{\xi'},x_n) = (L_0^+(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))^{-1}(\frac{\partial}{\partial x_n} + <\xi'>)^{l_0}(\varepsilon\frac{\partial}{\partial x_n} + <\varepsilon\xi'>)^{l_1-l_0}$$

$$\pi_+(L_0^-(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))^{-1}(\frac{\partial}{\partial x_n} + <\xi'>)^{-l_0}(\varepsilon\frac{\partial}{\partial x_n} + <\varepsilon\xi'>)^{l_0-l_1}lf(\widehat{\xi'},x_n)$$

is a solution of (3.2.39). Indeed, with

$$c(\varepsilon,\widehat{\xi'},\xi_n) \overset{\text{def}}{=} L_0^-(\varepsilon,\widehat{\xi'},\xi_n)(i\xi_n+<\xi'>)^{l_0}(i\varepsilon\xi_n+<\varepsilon\xi'>)^{l_1-l_0},$$

one obtains

$$\pi_+L_0(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})w_+ = \pi_+c(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})\pi_+(c(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))^{-1}\ lf(\widehat{\xi'},x_n) =$$

$$= f(\widehat{\xi'},x_n) + \pi_+c(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})v$$

with

$$v = (\pi_+-\mathrm{Id})\ \ (c(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))^{-1}lf.$$

Since l_0 and l_1-l_0 are nonnegative integers, the symbol c is analytic for $\mathrm{Im}\ \xi_n > 0$ and

$$\pi_+\ c(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})v = 0 \qquad \forall x_n > 0$$

holds because $\mathrm{supp}\ v \subset \mathbb{R}_-$. As a consequence of (3.2.41), (3.2.47), one finds that

$$\pi_+L_0^-(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})(\frac{\partial}{\partial x_n} + <\xi'>)^{-l_0}(\varepsilon\frac{\partial}{\partial x_n} + <\varepsilon\xi'>)^{l_0-l_1}lf \in \overset{\circ}{H}{}^{+}_{(s_1,s_2-r_2+l_0,s_3-r_3+l_1-l_0),\xi'}$$

Since $L_0^+(\varepsilon,\widehat{\xi'},\xi_n)^{-1}(i\xi_n+ <\xi'>)^{l_0}\ (i\ \varepsilon\ \xi_n+<\varepsilon\xi'>)^{l_1-l_0}$ is analytic for $\mathrm{Im}\ \xi_n < 0$, one has $w_+ \in \overset{\circ}{H}{}^{+}_{(s),\xi'}$. This ends the proof of Theorem 3.2.9. $\square$

3.3. Reduction to regular perturbations

Let $L(\varepsilon,\xi) \in D^{k_0}_{(\nu_1,2r_2,2r_3)}$, $L_j(\varepsilon,\xi) \in \mathcal{L}_{\mu_j}$, $1 \le j \le l_1$, be such that their principal parts L_0, L_{j0} satisfy the assumptions of Theorem 3.2.1 or Theorem 3.2.9. Consider the following singularly perturbed boundary value problem in $\mathbb{R}_+$ with the parameters $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1}$:

$$(3.3.1) \qquad \pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, u_+(x_n) = f(x_n) \quad , \ x_n > 0$$

$$(3.3.2) \qquad \pi_0 L_j(\varepsilon,\xi',-i\frac{\partial}{\partial x_n}) \quad u_+ = \phi_j \qquad , \ 1 \le j \le l_1$$

where the solution $u(x_n)$ is sought in the class of functions decreasing for $x_n \to \infty$. With (3.3.1), (3.3.2) are associated the Wiener-Hopf operators

$$\mathfrak{A}^{\varepsilon}(\xi') = \begin{pmatrix} \pi_+ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L_1(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L_{l_1}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \end{pmatrix} , \quad \mathfrak{A}^{\varepsilon}_0(\xi') = \begin{pmatrix} \pi_+ L_0(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L_{10}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L_{l_1 0}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})\, l_0 \end{pmatrix}$$

Let $L^0(\xi)$, $L^0_j(\xi)$ be the reduced symbols of $L(\varepsilon,\xi)$ and $L_j(\varepsilon,\xi)$, respectively, and let the number l_0, $1 \le l_0 \le l_1$, be as in section 3.2.

The reduced problem to (3.3.1), (3.3.2) is given as follows:

$$(3.3.3) \qquad \pi_+ L^0(\xi',-i\frac{\partial}{\partial x_n}) \quad u_+(x_n) = f(x_n), \quad x_n > 0$$

$$(3.3.4) \qquad \pi_0 L^0_j(\xi',-i\frac{\partial}{\partial x_n}) \quad u_+ = \phi_j, \quad 1 \le j \le l_0.$$

$$\mathfrak{A}^{0}(\xi') = \begin{pmatrix} \pi_+ L^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L^0_1(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L^0_{l_0}(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \end{pmatrix} , \quad \mathfrak{A}^{0}_0(\xi') = \begin{pmatrix} \pi_+ L^0_0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L^0_{10}(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L^0_{l_0 0}(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \end{pmatrix}$$

Assume first that the principal parts L_0, L_{j0} of L, L_j satisfy the conditions of Theorem 3.2.1. Let the spaces $H_{\xi'}$, $K_{\xi'}$ of the solutions and of the data to the perturbed problem, respectively, be given by

$$H_{\xi'} \overset{\text{def}}{=} H_{(s),\xi'}(\mathbb{R}_+)$$

$$K_{\xi'} \overset{\text{def}}{=} H_{(s-\nu),\xi'}(\mathbb{R}_+) \times \prod_{1\le j\le r_2} \mathbb{C}_{(\tau_j),\xi'} \times \prod_{r_2<j\le r_2+r_3} \mathbb{C}_{(\sigma_j),\xi'}.$$

Moreover, we introduce the spaces $H^0_{\xi'}$, $K^0_{\xi'}$, by

$$H^0_{\xi'} \overset{\text{def}}{=} H_{(s),\xi'}(\mathbb{R}_+)$$

$$K^0_{\xi'} \overset{\text{def}}{=} H_{(s-\nu_2 e_2),\xi'}(\mathbb{R}_+) \times \prod_{1\le j\le r_2} \mathbb{C}_{(s-(m_j+\frac{1}{2})e_2),\xi'}$$

Since the reduced problem satisfies the coerciveness condition, the Wiener-Hopf operator $(\mathfrak{A}^0_0(\widehat{\xi'}))^{-1}$ exists for $\forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$. Moreover, for $\varepsilon \in (0,\varepsilon_0]$ let $(\mathfrak{A}^\varepsilon_0(\widehat{\xi'}))^{-1}$ be the inverse operator of $\mathfrak{A}^\varepsilon_0(\widehat{\xi'})$ constructed in section 3.2. For $\forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, we define the Wiener-Hopf operators R^ε_r and R^ε_l, the so-called right and left reducing operators, by

$$\text{(3.3.5)} \qquad \begin{aligned} R^\varepsilon_r(\xi') &\overset{\text{def}}{=} \mathrm{Id} \\ R^\varepsilon_l(\xi') &\overset{\text{def}}{=} \mathfrak{A}^\varepsilon_0(\widehat{\xi'})\ (\mathfrak{A}^0_0(\widehat{\xi'}))^{-1} \end{aligned}$$

The inverse operators S^ε_r, S^ε_l to R^ε_r, R^ε_l, are given by

$$\text{(3.3.6)} \qquad \begin{aligned} S^\varepsilon_r(\xi') &= \mathrm{Id} \\ S^\varepsilon_l(\xi') &= \mathfrak{A}^0_0(\widehat{\xi'})(\mathfrak{A}^\varepsilon_0(\widehat{\xi'}))^{-1} \end{aligned}$$

Assume now that the conditions of Theorem 3.2.9 are satisfied. Let the spaces $H_{\xi'}$, $K_{\xi'}$ of the solutions and of the data to the perturbed problem, respectively, be given by

$$H_{\xi'} \overset{\text{def}}{=} \overset{\circ}{H}{}^{+}_{(s),\xi'}$$

$$K_{\xi'} \overset{\text{def}}{=} H_{(s-\nu),\xi'}(\mathbb{R}_+) \times \prod_{1\le j\le l_0} \mathbb{C}_{(\tau_j),\xi'} \times \prod_{l_0<j\le l_1} \mathbb{C}_{(\sigma_j),\xi'} .$$

Moreover, let

$$H_\xi^0 \overset{\text{def}}{=} \overset{\circ}{H}{}^{+}_{s_2,\xi'}$$

$$K_\xi^0 \overset{\text{def}}{=} H_{s_2-\nu_2,\xi'}(\mathbb{R}_+) \times \prod_{1\le j\le l_0} \mathbb{C}_{s_2-(m_j+\frac{1}{2}),\xi'}$$

In this case, we define the Wiener-Hopf operators R_r^ε and R_1^ε for $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ as follows:

$$(3.3.5') \qquad \begin{aligned} R_r^\varepsilon(\xi') &= \varepsilon^{-s_1}\left(\varepsilon\frac{\partial}{\partial x_n} + \langle\varepsilon\xi'\rangle\right)^{s_3} \\ R_1^\varepsilon(\xi') &= \mathfrak{A}_0^\varepsilon(\hat{\xi'})(R_r^\varepsilon(\xi'))^{-1}(\mathfrak{A}_0^0(\hat{\xi'}))^{-1} \end{aligned}$$

The inverse operators S_r^ε, S_1^ε to R_r^ε, R_1^ε be given by

$$(3.3.6') \qquad \begin{aligned} S_r^\varepsilon(\xi') &= \varepsilon^{s_1}\left(\varepsilon\frac{\partial}{\partial x_n} + \langle\varepsilon\xi'\rangle\right)^{-s_3} \\ S_1^\varepsilon(\xi') &= \mathfrak{A}_0^0(\hat{\xi'})\; R_r^\varepsilon(\xi')\; (\mathfrak{A}_0^\varepsilon(\hat{\xi'}))^{-1}. \end{aligned}$$

Let $s \in \mathbb{R}^3$ be such that the conditions in section 3.2 are satisfied and let $\gamma_0 = \min(1, \beta+\frac{1}{2}-s_2)$.

Then one has:

<u>Theorem 3.3.1.</u> Let $\mathfrak{A}^\varepsilon(\xi')$ be a coercive singularly perturbed Wiener-Hopf operator on the half line. With $R_r^\varepsilon, R_1^\varepsilon, S_r^\varepsilon, S_1^\varepsilon$ defined by (3.3.5), (3.3.6), (3.3.5'), (3.3.6'), the diagram

$$\text{(3.3.7)}\qquad \begin{tikzcd}[row sep=large, column sep=huge] H_{\xi'} \arrow[r, "\mathcal{A}^{\varepsilon}(\xi')"] \arrow[d, shift left, "R_r^{\varepsilon}"] & K_{\xi'} \arrow[d, shift right, "S_1^{\varepsilon}(\xi')"'] \\ H^0_{\xi'} \arrow[r, "\mathcal{A}^0(\xi')"] \arrow[u, shift left, "S_r^{\varepsilon}"] & K^0_{\xi'} \arrow[u, shift right, "R_1^{\varepsilon}(\xi')"'] \end{tikzcd}$$

is commutative modulo operators of a norm bounded by $C\varepsilon^{\gamma_0}|\ln \varepsilon|$ with some constant $C > 0$. In other words, the following inequalities hold for $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $\varepsilon \in (0,\varepsilon_0]$:

$$\text{(3.3.8)}\qquad ||\mathcal{A}^{\varepsilon}(\xi') - R_1^{\varepsilon}(\xi')\mathcal{A}^0(\xi')\, R_r^{\varepsilon}(\xi')||_{\mathrm{Hom}(H_{\xi'},K_{\xi'})} \le C\varepsilon^{\gamma_0}|\ln \varepsilon|$$

$$\text{(3.3.9)}\qquad \begin{cases} ||R_r^{\varepsilon}(\xi')\, S_r^{\varepsilon}(\xi') - \mathrm{Id}||_{\mathrm{Hom}\,(H^0_{\xi'},H^0_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \\ ||R_1^{\varepsilon}(\xi')\, S_1^{\varepsilon}(\xi') - \mathrm{Id}||_{\mathrm{Hom}\,(K_{\xi'},K_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \end{cases}$$

$$\text{(3.3.10)}\qquad \begin{cases} ||S_r^{\varepsilon}(\xi')\, R_r^{\varepsilon}(\xi') - \mathrm{Id}||_{\mathrm{Hom}(H_{\xi'},H_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \\ ||S_1^{\varepsilon}(\xi')\, R_1^{\varepsilon}(\xi') - \mathrm{Id}||_{\mathrm{Hom}(K^0_{\xi'},K^0_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \end{cases}$$

$$\text{(3.3.11)}\qquad ||\mathcal{A}^0(\xi') - S^{\varepsilon}(\xi')\mathcal{A}^{\varepsilon}(\xi')\, S_r^{\varepsilon}(\xi')||_{\mathrm{Hom}(H^0_{\xi'},K^0_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon|$$

where the constant C does not depend upon ε, ξ'.

Remark 3.3.2. In the special case of the linear plate problem, Theorem 3.3.1 was proved in [Fr-W]. Moreover, in [Fr-W], the commutativity of (3.3.7) was stated without proof for singularly perturbed differential boundary value problems.

Proof of Theorem 3.3.1. Since the symbol of $\mathcal{A}^{\varepsilon}$ does not depend upon x_n, the definitions of R_r^{ε}, R_1^{ε}, S_r^{ε}, S_1^{ε} imply that for $\forall\ \varepsilon \in (0,\varepsilon_0]$, $\forall\ \xi' \in \mathbb{R}^{n-1}\setminus\{0\}$:

$$R_1^\varepsilon(\xi')\; S_1^\varepsilon(\xi') = \mathrm{Id}_{K_{\xi'}}\;,\; S_1^\varepsilon(\xi')\; R_1^\varepsilon(\xi') = \mathrm{Id}_{K^0_{\xi'}}$$

$$R_r^\varepsilon(\xi')\; S_r^\varepsilon(\xi') = \mathrm{Id}_{H^0_{\xi'}}\;,\; S_r^\varepsilon(\xi')\; R_r^\varepsilon(\xi') = \mathrm{Id}_{H_{\xi'}}$$

This proves (3.3.9), (3.3.10). Now (3.3.8) will be proved under the additional assumptions $p_j \geq 0$, $\gamma_j = 0$ for $1 \leq j \leq l_1$ and $\nu_1 = 0$.

Let the Wiener-Hopf operators $\mathfrak{A}_1^0$, $\mathfrak{A}_{01}^0$ be defined as follows:

$$\mathfrak{A}_1^0(\xi') = \begin{pmatrix} \pi_+ L^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L_1^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L_{l_0}^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} l_1-l_0 \end{pmatrix} ,\; \mathfrak{A}_{01}^0(\xi') = \begin{pmatrix} \pi_+ L_0^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \pi_0 L_{10}^0(\xi'-i\frac{\partial}{\partial x_n})\, l_0 \\ \vdots \\ \pi_0 L_{l_0 0}^0(\xi',-i\frac{\partial}{\partial x_n})\, l_0 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} l_1-l_0 \end{pmatrix}$$

and let

$$(3.3.12)\qquad \begin{cases} \mathfrak{A}_p^\varepsilon(\xi') \overset{\text{def}}{=} \mathfrak{A}^\varepsilon(\xi') - \mathfrak{A}_1^0(\xi') \\ \mathfrak{A}_{0p}^\varepsilon(\xi') \overset{\text{def}}{=} \mathfrak{A}_0^\varepsilon(\xi') - \mathfrak{A}_{01}^0(\xi') \end{cases}$$

Obviously, one has

$$\mathfrak{A}_{01}^0(\widehat{\xi'})\; \mathfrak{A}_0^0(\widehat{\xi'})^{-1} = \binom{\mathrm{Id}}{0}$$

where $\mathrm{Id} \in \mathrm{Hom}(\mathbb{C}^{l_0+1}, \mathbb{C}^{l_0+1})$ is the identity matrix and $0 \in \mathrm{Hom}(\mathbb{C}^{l_0+1}, \mathbb{C}^{l_1-l_0})$ is the zero matrix.

Therefore, the product $R_1^\varepsilon\, \mathfrak{A}^0\, R_r^\varepsilon$ can be rewritten as follows:

$$(3.3.13) \qquad R_1^\varepsilon(\xi')\,\mathcal{A}^0(\xi')\,R_r^\varepsilon(\xi') =$$

$$= \mathcal{A}_1^0(\xi') + \mathcal{A}_{0p}^\varepsilon(\hat{\xi}')[\mathrm{Id}+S_r^\varepsilon(\xi')(\mathcal{A}_0^0(\hat{\xi}'))^{-1}(\mathcal{A}^0(\xi')-\mathcal{A}_0^0(\hat{\xi}'))\,R_r^\varepsilon(\xi')]$$

$$= (\mathcal{A}_1^0(\xi')+\mathcal{A}_p^\varepsilon(\xi'))+(\mathcal{A}_p^\varepsilon(\hat{\xi}')-\mathcal{A}_p^\varepsilon(\xi'))+(\mathcal{A}_{0p}^\varepsilon(\hat{\xi}')-\mathcal{A}_p^\varepsilon(\hat{\xi}'))+$$

$$\mathcal{A}_{0p}^\varepsilon(\hat{\xi}')S_r^\varepsilon(\xi')(\mathcal{A}_0^0(\hat{\xi}'))^{-1}(\mathcal{A}^0(\xi')-\mathcal{A}_0^0(\hat{\xi}'))\,R_r^\varepsilon(\xi')$$

$$= \mathcal{A}^\varepsilon(\xi') + \sum_{j=1}^{3} Q_j.$$

Since ν_3, $p_j \geq 0$, Lemma 3.1. 7 implies that

$$(3.3.14) \qquad L_p(\varepsilon,\xi) \overset{\mathrm{def}}{=} L(\varepsilon,\xi) - L^0(\xi) \in \mathcal{D}^{k_0}_{(-1,\nu_2+1,\nu_3-1)}$$

$$(3.3.15) \qquad L_{jp}(\varepsilon,\xi) \overset{\mathrm{def}}{=} L_j(\varepsilon,\xi) - L_j^0(\xi) \in \mathcal{L}_{(-1,m_j+1,p_j-1)}, \quad 1 \leq j \leq l_1.$$

Therefore, (3.1.5) yields:

$$L_p(\varepsilon,\hat{\xi}',\xi_n) - L_p(\varepsilon,\xi) \in \mathcal{D}^{k_0}_{(-1,\nu_2,\nu_3)}$$

$$L_{jp}(\varepsilon,\hat{\xi}',\xi_n) - L_{jp}(\varepsilon,\xi) \in \mathcal{L}_{(-1,m_j,p_j)} \quad , \quad 1 \leq j \leq l_1$$

Therefore, the norm in $\mathrm{Hom}(\mathcal{H}_{\xi'},\mathcal{K}_{\xi'})$ of the operator

$$Q_1 = \mathcal{A}_p^\varepsilon(\hat{\xi}')-\mathcal{A}_p^\varepsilon(\xi') = \begin{pmatrix} \pi_+(L_p(\varepsilon,\hat{\xi}',-i\frac{\partial}{\partial x_n})-L_p(\varepsilon,\xi',-i\frac{\partial}{\partial x_n}))1_0 \\ \pi_0(L_{1p}(\varepsilon,\hat{\xi}',-i\frac{\partial}{\partial x_n})-L_{1p}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n}))1_0 \\ \vdots \\ \pi_0(L_{l_1p}(\varepsilon,\hat{\xi}',-i\frac{\partial}{\partial x_n})-L_{l_1p}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n}))1_0 \end{pmatrix}$$

is bounded by $C\,\varepsilon$, where the constant C does not depend upon ε,ξ'.

Applying again (3.3.14), (3.3.15), one finds that

$$L_{0p}(\varepsilon,\hat{\xi}',\xi_n) - L_p(\varepsilon,\hat{\xi}',\xi_n) \in \mathcal{D}^{k_0}_{(-1,\nu_2,\nu_3)}$$

$$L_{j0p}(\varepsilon,\hat{\xi}',\xi_n) - L_{jp}(\varepsilon,\hat{\xi}',\xi_n) \in \mathcal{L}_{(-1,m_j,p_j)}, \quad 1 \leq j \leq l_1.$$

Therefore, the norm in Hom $(H_{\xi'}, K_{\xi'})$ of the operator

$$Q_2 = \mathcal{O}\!\mathcal{L}^{\varepsilon}_{0p}(\widehat{\xi'}) - \mathcal{O}\!\mathcal{L}^{\varepsilon}_{p}(\widehat{\xi'}) = \begin{pmatrix} \pi_+(L_{0p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) - L_p(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))1_0 \\ \pi_0(L_{10p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) - L_{1p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))1_0 \\ \vdots \\ \pi_0(L_{1_10p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}) - L_{1_10p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n}))1_0 \end{pmatrix}$$

is bounded by C ε, where the constant C does not depend upon ε, ξ'.

Now we are going to estimate the norm of the operator

$$Q_3 = \mathcal{O}\!\mathcal{L}^{\varepsilon}_{0p}(\widehat{\xi'})S^{\varepsilon}_r(\xi')(\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))^{-1}(\mathcal{O}\!\mathcal{L}^0(\xi')-\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))\ R^{\varepsilon}_r(\xi').$$

Since $\mathcal{O}\!\mathcal{L}^0(\xi') - \mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'})$ is an operator of lower order compared to $\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'})$, one has

$$(\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))^{-1}(\mathcal{O}\!\mathcal{L}^0(\xi')-\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))\in \mathrm{Hom}(\overset{\circ}{H}{}^+_{s_2,\xi'},\ \overset{\circ}{H}{}^+_{s_2+1,\xi'}) \cap$$
$$\cap\ \mathrm{Hom}(H_{(s),\xi'}(\mathbb{R}_+),\ H_{(s+e_2),\xi'}(\mathbb{R}_+))$$

and

$$S^{\varepsilon}_r(\xi')(\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))^{-1}(\mathcal{O}\!\mathcal{L}^0(\xi')-\mathcal{O}\!\mathcal{L}^0_0(\widehat{\xi'}))\ R^{\varepsilon}_r(\xi') \tag{3.3.16}$$
$$\in \mathrm{Hom}(H_{(s),\xi'}(\mathbb{R}_+),\ H_{(s+e_2),\xi'}(\mathbb{R}_+))\ .$$

Moreover, one has:

$$\mathcal{O}\!\mathcal{L}^{\varepsilon}_{0p}(\widehat{\xi'}) = \begin{pmatrix} \pi_+L_{0p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 \\ \pi_0L_{10p}(\varepsilon,\widehat{\xi'},\ -i\frac{\partial}{\partial x_n})1_0 \\ \vdots \\ \pi_0L_{1_00p}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 \\ \pi_0L_{1_0+10}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 \\ \vdots \\ \pi_0L_{1_10}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 \end{pmatrix}$$

where L_{0p}, L_{j0p}, $1 \le j \le l_0$, are the principal parts of the symbols defined in (3.3.14), (3.3.15), respectively. The inclusions $L_{0p} \in \mathcal{D}^{k_0}_{(-1,\nu_2+1,\nu_3)}$, $L_{j0p} \in \mathcal{L}_{(-1,m_j+1,p_j)}$ yield:

$$||\pi_+ L_{0p}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})l_0 u||^+_{(s-\nu),\xi'} \le C||u||^+_{(s-\nu+(-1,\nu_2+1,\nu_3)),\xi'} =$$

$$= C\ \varepsilon\ ||u||^+_{(s+e_2),\xi'}$$

$$[\pi_0 L_{j0p}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})l_0 u]_{(\tau_j),\xi'} \le C||\pi_+ L_{j0p}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})l_0 u||^+_{(s-(0,m_j,p_j)),\xi'}$$

$$\le C||u||^+_{(s+e_2),\xi'}\ ,\quad 1 \le j \le l_0$$

$$\text{for } \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}\ \forall u \in H_{(s+e_2),\xi'}(\mathbb{R}_+)$$

with a constant C which does not depend upon ε,ξ',u.

Using the trace and embedding theorems established in [Fr 1] (see also section 2 of this paper for the formulation of these results), it will be shown that for $l_0 < j \le l_1$:

$$(3.3.17)\qquad [\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})l_0 u]_{(\sigma_j),\xi'} \le C\ \varepsilon^{\gamma_0}|\ln\varepsilon|\ ||u||^+_{(s+e_2),\xi'}$$

Consider first the case $s_2-m_j+1 > \frac{1}{2}$. Then one has:

$$[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\tfrac{\partial}{\partial x_n})l_0 u]_{(\sigma_j),\xi'} = \varepsilon^{m_j+\frac{1}{2}-s_2}[\pi_0 L_{j0} l_0 u]_{(s_1,0,s_2+s_3-p_j-\frac{1}{2}),\xi'}$$

$$\le C\ \varepsilon^{m_j+\frac{1}{2}-s_2}[\pi_0 L_{j0} l_0 u]_{(s_1,s_2-m_j+\frac{1}{2},s_3-p_j),\xi'}$$

$$\le C\ \varepsilon^{m_{l_0+1}+\frac{1}{2}-s_2}\ ||\pi_+ L_{j0} l_0 u||^+_{(s_1,s_2-m_j+1,s_3-p_j),\xi'}$$

$$\le C\ \varepsilon^{\gamma_0}\ ||u||^+_{(s+e_2),\xi'}$$

with C independent upon ε,ξ',u. In the case $s_2-m_j+1 = \frac{1}{2}$, one obtains

$$[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 u]_{(\sigma_j),\xi'} = \varepsilon^{m_j+\frac12-s_2}[\pi_0 L_{jo}1_0 u]_{(s_1,0,s_3-p_j),\xi'}$$

$$\le C\,\varepsilon\,|\ln\varepsilon|\;||\pi_+ L_{j0}1_0 u||^+_{(s_1,\frac12,s_3-p_j),\xi'}$$

$$\le C\,\varepsilon\,|\ln\varepsilon|\;||\pi_+ L_{j0}1_0 u||^+_{(s_1,s_2-m_j+1,s_3-p_j),\xi'}$$

$$\le C\,\varepsilon\,|\ln\varepsilon|\;||u||^+_{(s_1,s_2+1,s_3),\xi'}$$

Finally, for $s_2-m_j+1 < \frac12$, one has:

$$[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 u]_{(\sigma_j),\xi'} \le$$

$$\le \varepsilon\;[\pi_0 L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 u]_{(s_1+(s_2+1-m_j)-\frac12,0,(s_2+1-m_j)+s_3-p_j-\frac12)}$$

$$\le C\,\varepsilon\;||\pi_+ L_{j0}(\varepsilon,\widehat{\xi'},-i\frac{\partial}{\partial x_n})1_0 u||^+_{(s_1,s_2+1-m_j,s_3-p_j),\xi'}$$

$$\le C\,\varepsilon\;||u||^+_{(s+e_2),\xi'}$$

Therefore, (3.3.17) is proved and one has:

$$||\mathcal{A}^{\varepsilon}_{Op}(\widehat{\xi'})||_{Hom(H_{(s+e_2),\xi'}(\mathbb{R}_+),\,K_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln\varepsilon|$$

where C does not depend upon ε,ξ'. As an immediate consequence of (3.3.16) and of the last estimate, one obtains (3.3.8).

Now we drop the assumptions $p_j \ge 0$, $\gamma_j = 0$, $1 \le j \le l_1$ and $\nu_1 = 0$. Let $p'_j \overset{def}{=} \min(p_j,0)$ for $1 \le j \le l_1$ and let the Wiener-Hopf operator $\mathcal{B}^{\varepsilon}$ be defined by

$$\mathcal{B}^{\varepsilon}(\xi') = \mathrm{diag}(\varepsilon^{-\nu_1}Id,\varepsilon^{-\gamma_1}<\varepsilon\xi'>^{p'_1},\ldots,\varepsilon^{-\gamma_{l_1}}<\varepsilon\xi'>^{p'_{l_1}})$$

Since $p'_j \le 0$ implies that $\varepsilon^{\gamma_j}<\varepsilon\xi'>^{-p'_j} \in L_{(-\gamma_j,0,-p'_j)}(\mathbb{R}^n)$, one has

$\varepsilon^{\gamma_j}\langle\varepsilon\xi'\rangle^{-p_j'}L_{j0}(\varepsilon,\xi\ ) \in L_{(0,m_j,\max(p_j,0))}$ for $1 \le j \le l_1$.

The operator

$$\widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}(\xi') \overset{\mathrm{def}}{=} B^{\varepsilon}(\xi')^{-1}\mathcal{O}\!\mathcal{L}^{\varepsilon}(\xi') = \begin{pmatrix} \pi_+ \ \varepsilon^{\nu_1} \ L(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})l_0 \\ \pi_0 \ \varepsilon^{\gamma_1}\langle\varepsilon\xi'\rangle^{-p_1'} L_1(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})l_0 \\ \vdots \\ \pi_0 \ \varepsilon^{\gamma_{l_1}}\langle\varepsilon\xi'\rangle^{-p_{l_1}'} L_{l_1}(\varepsilon,\xi',-i\frac{\partial}{\partial x_n})l_0 \end{pmatrix}$$

satisfies the coerciveness condition. Therefore, (3.3.8) holds for $\mathcal{O}\!\mathcal{L}^{\varepsilon}$ replaced by $\widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}$ and R_l^{ε}, R_r^{ε} replaced by the operators

$$\tilde{R}_l^{\varepsilon}(\xi') \overset{\mathrm{def}}{=} (B^{\varepsilon}(\hat{\xi'}))^{-1} R_l^{\varepsilon}(\xi')$$

$$\tilde{R}_r^{\varepsilon}(\xi') \overset{\mathrm{def}}{=} R_r^{\varepsilon}(\xi')$$

One has:

$$\begin{aligned}\mathcal{O}\!\mathcal{L}^{\varepsilon}(\xi') &= B^{\varepsilon}(\xi')\widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}(\xi') \\ &= (B^{\varepsilon}(\hat{\xi'}) + (B^{\varepsilon}(\xi') - B^{\varepsilon}(\hat{\xi'})))\ \widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}(\xi') \\ &= B^{\varepsilon}(\hat{\xi'})\ \tilde{R}_l^{\varepsilon}(\xi')\ \mathcal{O}\!\mathcal{L}^{0}(\xi')\ \tilde{R}_r^{\varepsilon} + Q_1 + Q_2 \\ &= R_l^{\varepsilon}(\xi')\ \mathcal{O}\!\mathcal{L}^{0}(\xi')\ R_r^{\varepsilon}(\xi') + Q_1 + Q_2\end{aligned}$$

where

$$Q_1 = B^{\varepsilon}(\hat{\xi'})\ (\widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}(\xi') - \tilde{R}_l^{\varepsilon}(\xi')\ \mathcal{O}\!\mathcal{L}^{0}(\xi')\ \tilde{R}_r^{\varepsilon}(\xi'))$$

and

$$Q_2 = (B^{\varepsilon}(\xi') - B^{\varepsilon}(\hat{\xi'}))\ \widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}(\xi').$$

The estimate (3.3.8) with $\mathcal{O}\!\mathcal{L}^{\varepsilon}$, R_r^{ε}, R_l^{ε} replaced by $\widetilde{\mathcal{O}\!\mathcal{L}}^{\varepsilon}$, $\tilde{R}_r^{\varepsilon}$, $\tilde{R}_l^{\varepsilon}$ yields that

$$||Q_1||_{Hom(H_{\xi'},K_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln\varepsilon|$$

holds uniformly with respect to ε,ξ'. Since

$$\varepsilon^{-\gamma_j}(\langle\varepsilon\xi'\rangle^{p_j'} - \langle\varepsilon\xi'\rangle^{p_j'}) \in L_{(\gamma_j-1,0,p_j')}(\mathbb{R}^{n-1}),$$

one has

$$||Q_2||_{Hom(H_{\xi'},K_{\xi'})} \le C\,\varepsilon.$$

This proves (3.3.8). The estimate (3.3.11) is an immediate consequence of (3.3.8), (3.3.9), (3.3.10). □

Corollary 3.3.3. Assume that the inverse operator $(\mathfrak{A}^0(\xi'))^{-1}$ of $\mathfrak{A}^0(\xi')$ exists for $\forall\xi' \in \mathbb{R}^{n-1}$ and that it is bounded from $K^0_{\xi'}$ into $H^0_{\xi'}$ uniformly with respect to $\xi' \in \mathbb{R}^{n-1}$. Then the inverse operator $(\mathfrak{A}^\varepsilon(\xi'))^{-1} \in$ $\in Hom(K_{\xi'},H_{\xi'})$ exists for $\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ with ε_0 sufficiently small, and it can be expanded in a convergent series as follows:

$$\mathfrak{A}^\varepsilon(\xi')^{-1} = S^\varepsilon_r \sum_{k\ge 0}[(\mathfrak{A}^0(\xi'))^{-1}(\mathfrak{A}^0(\xi')-S^\varepsilon_1(\xi')\mathfrak{A}^\varepsilon(\xi')S^\varepsilon_r(\xi'))]^k \cdot (\mathfrak{A}^0(\xi'))^{-1} S^\varepsilon_1(\xi') \tag{3.3.18}$$

Proof. The equation $\mathfrak{A}^\varepsilon\,U = F$ can be rewritten as follows:

$$S^\varepsilon_1\,\mathfrak{A}^\varepsilon S^\varepsilon_r((S^\varepsilon_r)^{-1}\,U) = S^\varepsilon_1 F$$

$$\mathfrak{A}^0(Id + (\mathfrak{A}^0)^{-1}(S^\varepsilon_1\,\mathfrak{A}^\varepsilon S^\varepsilon_r-\mathfrak{A}^0))((S^\varepsilon_r)^{-1}U) = S^\varepsilon_1 F$$

Since (3.3.11) implies that

$$||(\mathfrak{A}^0)^{-1}(S^\varepsilon_1\mathfrak{A}^\varepsilon S^\varepsilon_r-\mathfrak{A}^0)||_{Hom(H^0_{\xi'},H^0_{\xi'})} \le C\,\varepsilon^{\gamma_0}|\ln\varepsilon|$$

holds with C independent upon ε,ξ', the inverse of the operator

$\mathrm{Id} + (\mathcal{O}\!\mathcal{L}^0)^{-1}(S_1^\varepsilon \mathcal{O}\!\mathcal{L}^\varepsilon S_r^\varepsilon - \mathcal{O}\!\mathcal{L}^0)$ exists for $\varepsilon \in (0,\varepsilon_0]$, ε_0 sufficiently small, and can be written as a Neumann series. This leads to the formula (3.3.18). $\square$

Remark 3.3.4. Let $\gamma \in (0,\gamma_0)$ be fixed. In [Fr-W], the following procedure was indicated (for the differential case) in order to construct high-order approximations for the solution of (3.3.1), (3.3.2). With $S^\varepsilon(\xi') \overset{\text{def}}{=} \mathcal{O}\!\mathcal{L}_0^0(\hat{\xi}')(\mathcal{O}\!\mathcal{L}_0^\varepsilon(\hat{\xi}'))^{-1}$, let the functions $u_\varepsilon^{(l)} \in H_{\xi'}$, $l \geq 0$, be recursively defined by

$$\mathcal{O}\!\mathcal{L}^0 u_\varepsilon^{(l)} = \varepsilon^{-l\gamma} S^\varepsilon((f,\phi_1,\ldots,\phi_{r_2+r_3})^\tau - \sum_{0\leq l'\leq l-1} \varepsilon^{\gamma l'} \mathcal{O}\!\mathcal{L}^\varepsilon u_\varepsilon^{(l')}), \tag{3.3.19}$$

where the sum is zero for $l = 0$. Then one has:

$$||u_\varepsilon - \sum_{0\leq l\leq r} \varepsilon^{\gamma l} u_\varepsilon^{(l)}||^+_{(s),\xi'} \leq C\, \varepsilon^{\gamma(r+1)} ||(f,\phi_1,\ldots,\phi_{r_2+r_3})^\tau||_{K_{\xi'}}$$

where the constant C does not depend upon $\varepsilon,\nu,f,\phi_1,\ldots,\phi_{r_2+r_3}$. This procedure is equivalent to the construction of approximations for u_ε which is obtained using (3.3.18). Note that, as a consequence of (3.3.19), the functions $\varepsilon^{\gamma l} u_\varepsilon^{(l)}$, $l \geq 0$, do not depend upon γ. In the case of C^∞-data $f,\phi_1,\ldots,\phi_{r_2+r_3}$, the asymptotic expansion $\sum_{l\geq 0} \varepsilon^{\gamma l} u_\varepsilon^{(l)}$ for the solution can be split into a smooth part and a boundary layer.
This yields the classical Vishik-Lyusternik approximations which were obtained in [V-L] by the matching procedure.

It should be noted (though, probably, it has no consequences for practical computations) that the series $\sum_{l\geq 0} \varepsilon^{\gamma l}\, u_\varepsilon^{(l)}$ constructed above is convergent for $\varepsilon \in (0,\varepsilon_0]$, whereas the expansions in [V-L] for the smooth part and the boundary layer are only asymptotically convergent for $\varepsilon \downarrow 0$.

4.1. Definitions and auxiliary results.

Definition 4.1.1 ([Fr-W]). A function $L : \mathbb{R}^n \times (0,\varepsilon_0] \times \mathbb{R}^n \setminus S \to \mathbb{C}$, with S a set of measure zero, is said to belong to the class $\mathcal{L}_\nu$ for $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{R}^3$, if there exists a decomposition $L(x,\varepsilon,\xi) = L'(x,\varepsilon,\xi) + L_\infty(\varepsilon,\xi)$, such that

(i) the function L_∞ satisfies (3.1.4) - (3.1.8),

(ii) the function $\mathbb{R}^n \ni x \to L'(x,\varepsilon,\xi)$ is smooth for $\forall\, \varepsilon \in (0,\varepsilon_0]$, $\forall\, \xi \in \mathbb{R}^n \setminus S$, and for $\forall\, \alpha \in \mathbb{N}^n$, $\forall\, x \in \mathbb{R}^n$, the symbol $D_x^\alpha L'(x,\varepsilon,\xi)$ satisfies (3.1.4) - (3.1.8), where m is independent upon α, x and where $C = C(\alpha,x)$ satisfies the inequality

$$|C(\alpha,x)| \le C_{k,\alpha} \langle x\rangle^{-k} \quad \forall\, \alpha \in \mathbb{N}^n,\ \forall\, x \in \mathbb{R}^n,\ \forall\, k \ge 0$$

with $C_{k,\alpha}$ independent upon x.

For symbols $L \in \mathcal{L}_\nu$ depending on x, we define the pseudodifferential operator with the symbol L, the reduced operator and the (proper) ellipticity as in section 3.1.

Lemma 4.1.2. If $L \in \mathcal{L}_\nu$, then Op L has order ν.

Proof. Let $L = L' + L_\infty$ be the decomposition which exists according to Definition 4.1.1. Since $L_\infty \in \mathcal{L}_\nu$ is independent upon x, Op L_∞ has obviously order ν. Thus, it is sufficient to prove that Op L' has order ν. The inequalities

$$|D_x^\alpha L'(x,\varepsilon,\xi)| \le C_{\alpha,k}\varepsilon^{-\nu_1}\langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}\langle x\rangle^{-k} \quad \forall x,\xi \in \mathbb{R}^n,\ \forall\varepsilon \in (0,\varepsilon_0], \quad \forall\alpha \in \mathbb{N}^n\ \forall k > 0$$

yield

$$|\widehat{L'}(\zeta,\varepsilon,\xi)| \le C_k\varepsilon^{-\nu_1}\langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}\langle\zeta\rangle^{-k} \quad \forall\zeta,\xi \in \mathbb{R}^n,\ \forall\varepsilon \in (0,\varepsilon_0], \quad \forall k > 0 \tag{4.1.1}$$

where the hat denotes the Fourier transform with respect to the first variable. One has:

$$(\mathrm{Op}\ L'\ u)\hat{}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{L'}(\xi-\eta,\varepsilon,\eta)\hat{u}(\eta)\ d\eta\ .$$

Let $s \in \mathbb{R}^3$ be fixed and let $u \in H_{(s)}(\mathbb{R}^n)$. The norm in $H_{(s-\nu)}(\mathbb{R}^n)$ of the function Op L' u can be rewritten as follows:

$$||\mathrm{Op}\ L'u||_{(s-\nu)} = ||\varepsilon^{-(s_1-\nu_1)}\langle\xi\rangle^{s_2-\nu_2}\langle\varepsilon\xi\rangle^{s_3-\nu_3}(\mathrm{Op}\ L'u)\hat{}(\xi)||_{L^2(\mathbb{R}^n_\xi)} =$$

$$= ||\int K(\xi,\varepsilon,\eta)(\varepsilon^{-s_1}\langle\eta\rangle^{s_2}\langle\varepsilon\eta\rangle^{s_3}\hat{u}(\eta))d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

with

$$K(\xi,\varepsilon,\eta) \overset{\mathrm{def}}{=} \varepsilon^{\nu_1}\langle\xi\rangle^{s_2-\nu_2}\langle\varepsilon\xi\rangle^{s_3-\nu_3}\langle\eta\rangle^{-s_2}\langle\varepsilon\eta\rangle^{-s_3}\widehat{L'}(\xi-\eta,\varepsilon,\eta).$$

Since $||\varepsilon^{-s_1}\langle\eta\rangle^{s_2}\langle\varepsilon\eta\rangle^{s_3}\hat{u}(\eta)||_{L^2} = ||u||_{(s)}$, in order to prove the boundedness of Op L' from $H_{(s)}$ to $H_{(s-\nu)}$, it is sufficient to show that the integral operator K,

$$(K\ w)(\xi) \overset{\mathrm{def}}{=} \int_{\mathbb{R}^n} K(\xi,\varepsilon,\eta)\ w(\eta)d\eta\ ,$$

is bounded in $L^2(\mathbb{R}^n)$, uniformly with respect to ε. As a consequence of (4.1.1) and of the inequality

$$\langle\xi\rangle^t \langle\eta\rangle^{-t} \le C_t \langle\xi-\eta\rangle^{|t|} \quad \forall\xi,\eta \in \mathbb{R}^n,\ \forall t \in \mathbb{R}, \tag{4.1.2}$$

one obtains

$$|K(\xi,\varepsilon,\eta)| \le C_k\langle\xi-\eta\rangle^{m_1-k} \quad \forall\xi,\eta \in \mathbb{R}^n,\ \forall\varepsilon \in (0,\varepsilon_0],$$

where $m_1 = |s_2-\nu_2| + |s_3-\nu_3|$. Therefore, for k sufficiently large, namely for $k > m_1+n$, one has:

$$\int_{\mathbb{R}^n} |K(\xi,\varepsilon,\eta)|d\xi < C \tag{4.1.3}$$

$$\int_{\mathbb{R}^n} |K(\xi,\varepsilon,\eta)|d\eta < C \tag{4.1.4}$$

with C independent upon ξ,ε,η. These inequalities and Schur's Lemma on the boundedness of integral operators in L^2 (see [Sch]), implies that Op L' has order ν. □

<u>Definition 4.1.3</u>. A symbol $L \in \mathcal{L}_\nu$ is said to belong to the class $\mathcal{D}_\nu^k$ if there exists a decomposition $L(x,\varepsilon,\xi) = L'(x,\varepsilon,\xi) + L_\infty(\varepsilon,\xi)$ such that

(i) the function L_∞ satisfies the condition in Definition 3.1.15.

(ii) the function $\mathbb{R}^n \ni x \to L'(x,\varepsilon,\xi)$ is smooth for $\forall\ \varepsilon \in (0,\varepsilon_0]$, $\forall\ \xi \in \mathbb{R}^n \setminus S$, and for $\forall\ \alpha \in \mathbb{N}^n$, $\forall\ x \in \mathbb{R}^n$, the symbol $D_x^\alpha L'(x,\varepsilon,\xi)$ satisfies the condition in Definition 3.1.15, where $C = C(\alpha,x)$ satisfies the inequality

$$|C(\alpha,x)| \le C_{k,\alpha} \langle x\rangle^{-k} \quad \forall\alpha \in \mathbb{N}^n,\ \forall x \in \mathbb{R}^n,\ \forall k \ge 0 \tag{4.1.5}$$

with $C_{k,\alpha}$ independent upon x.

Lemma 4.1.4. Let $L(x,\varepsilon,\xi) \in \mathcal{D}^k_\nu$, $0 \le s_2 < k + \frac{1}{2}$, $s_2+s_3 \ge 0$.

Then one has:

$$(4.1.5) \qquad ||\pi_+ L(x,\varepsilon,-i\tfrac{\partial}{\partial x})\ l_0 u||^+_{(s-\nu)} \le C||u||^+_{(s)} \quad \forall u \in H_{(s)}(\mathbb{R}^n_+)$$

with a constant C which does not depend upon ε and u.

Proof. Let $l : H_{(s)}(\mathbb{R}^n_+) \to H_{(s)}(\mathbb{R}^n)$ be a bounded extension operator. Following [V-E], the norm of the function $\pi_+ L(x,\varepsilon,-i\frac{\partial}{\partial x}) l_0 u$ is estimated as follows:

$$||\pi_+ L(x,\varepsilon,-i\tfrac{\partial}{\partial x}) l_0 u||^+_{(s-\nu)} \le$$

$$\le ||\pi_+ L'(x,\varepsilon,-i\tfrac{\partial}{\partial x}) l_0 u||^+_{(s-\nu)} + ||\pi_+ L_\infty(\varepsilon,-i\tfrac{\partial}{\partial x}) l_0 u||^+_{(s-\nu)}$$

$$\le ||\pi_+ L'(x,\varepsilon,-i\tfrac{\partial}{\partial x}) l u||^+_{(s-\nu)} + ||\pi_+ L'(x,\varepsilon,-i\tfrac{\partial}{\partial x})(\pi_- l u)||^+_{(s-\nu)} +$$

$$+ ||\pi_+ L_\infty(\varepsilon,-i\tfrac{\partial}{\partial x}) l_0 u||^+_{(s-\nu)}$$

where $L = L' + L_\infty$ is the decomposition according to Definition 4.1.4. The Lemmas 4.1.2 and 3.1.17 yield:

$$||\pi_+ L'(x,\varepsilon,-i\tfrac{\partial}{\partial x}) l u||^+_{(s-\nu)} \le ||L'(x,\varepsilon,-i\tfrac{\partial}{\partial x}) l u||_{(s-\nu)} \le C||u||^+_{(s)}$$

and

$$||\pi_+ L_\infty(\varepsilon,-i\tfrac{\partial}{\partial x}) l_0 u||^+_{(s-\nu)} \le C||u||^+_{(s)},$$

respectively. Thus, we only have to show that

$$(4.1.6) \qquad ||\pi_+ L'(x,\varepsilon,-i\tfrac{\partial}{\partial x})(\pi_- l u)||^+_{(s-\nu)} \le C||u||^+_{(s)}.$$

For $s_2 \le k$, let

$$\varepsilon^{-(s_1-\nu_1)}(-i\xi_n+\langle\xi'\rangle)^{s_2-\nu_2}(-i\varepsilon\xi_n+\langle\varepsilon\xi'\rangle)^{s_3-\nu_3}L'(x,\varepsilon,\xi)=b'(x,\varepsilon,\xi)+r'(x,\varepsilon,\xi)$$

be the decomposition according to Definition 4.1.3. One has:

$$||\Pi^+_{\xi_n} L'(x,\varepsilon,-i\tfrac{\partial}{\partial x})(\pi\_lu)||^+_{(s-\nu)} =$$

$$= ||\Pi^+_{\xi_n}\, \varepsilon^{-(s_1-\nu_1)} (-i\xi_n+<\xi'>)^{s_2-\nu_2}(-i\varepsilon\xi_n+<\varepsilon\xi'>)^{s_3-\nu_3} \int \widehat{L'}(\xi-\eta,\varepsilon,\eta)\widehat{\pi\_lu}(\eta)d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

$$= ||\Pi^+_{\xi_n} \int s(\varepsilon,\xi,\eta)\ (\widehat{b'}(\xi-\eta,\varepsilon,\eta)+\widehat{r'}(\xi-\eta,\varepsilon,\eta))\widehat{\pi\_lu}(\eta)d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

where the function

$$s(\varepsilon,\xi,\eta) = \frac{(-i\xi_n+<\xi'>)^{s_2-\nu_2}\ (-i\varepsilon\xi_n+<\varepsilon\xi'>)^{s_3-\nu_3}}{(-i\eta_n+<\eta'>)^{s_2-\nu_2}\ (-i\varepsilon\eta_n+<\varepsilon\eta'>)^{s_3-\nu_3}}$$

satisfies the estimate

$$|s(\varepsilon,\xi,\eta)| \le C<\xi-\eta>^m$$

with $m = |s_2-\nu_2| + |s_3-\nu_3|$.

One has:

$$\Pi^+_{\xi_n} \int_{\mathbb{R}^n} s(\varepsilon,\xi,\eta)\ \widehat{b'}(\xi-\eta,\varepsilon,\eta)\ \widehat{\pi\_lu}(\eta)d\eta =$$

$$= \int \Pi^+_{\xi_n}\ s(\varepsilon,\xi,\xi-\zeta)\ \widehat{b'}(\zeta,\varepsilon,\xi-\zeta)\ \widehat{\pi\_lu}(\xi-\zeta)d\xi \equiv 0,$$

since the function under the π^+ sign is analytic for $\operatorname{Im}\xi_n > 0$. We estimate the remaining term as follows:

$$||\Pi^+_{\xi_n} \int\ s(\varepsilon,\xi,\eta)\ \widehat{r'}(\xi-\eta,\varepsilon,\eta)\quad \widehat{\pi\_lu}(\eta)d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

$$\le ||\int |s(\varepsilon,\xi,\eta)\ \widehat{r'}(\xi-\eta,\varepsilon,\eta)|\ |\widehat{\pi\_lu}(\eta)|d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

$$\le C_k\ ||\int <\xi-\eta>^{m-k}\varepsilon^{-s_1}<\eta'>^{s_2}<\varepsilon\eta'>^{s_3}|\widehat{\pi\_lu}(\eta)|d\eta||_{L^2(\mathbb{R}^n_\xi)}$$

$$\le C_{m+n+1} \| \varepsilon^{-s_1} \langle \eta' \rangle^{s_2} \langle \varepsilon\eta' \rangle^{s_3} \widehat{\pi\_lu}(\eta) \|_{L^2(\mathbb{R}^n_\eta)}$$

$$\le C_{m+n+1} \|u\|^+_{(s)}.$$

This ends the proof of Lemma 4.1.4 for $s_2 \le k$. The case $k < s_2 < k+\frac{1}{2}$ is treated in the same way as in the proof of Lemma 3.1.17, using the splitting of contour integrals as in [Fr 1]. □

Definition 4.1.5. A function $\mathbb{R}^{n-1} \times (0,\varepsilon_0] \times \mathbb{R}^n \ni (x',\varepsilon,\xi) \to k(x',\varepsilon,\xi)$ is said to be a Poisson symbol of order α, if there exists a decomposition $k(x',\varepsilon,\xi) = k'(x',\varepsilon,\xi) + k_\infty(\varepsilon,\xi)$, such that

(i) the function k_∞ satisfies the conditions in Definition 3.1.24,

(ii) the function $\mathbb{R}^{n-1} \ni x' \to k'(x',\varepsilon,\xi)$ is smooth for $\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi \in \mathbb{R}^n$, and for $\forall\alpha' \in \mathbb{N}^{n-1}$, $\forall x' \in \mathbb{R}^{n-1}$, the symbol $D^{\alpha'}_{x'} k'(x',\varepsilon,\xi)$ satisfies (3.1.24), where $C = C(\alpha',x')$ satisfies the inequality

$$|C(\alpha',x')| \le C_{k,\alpha'} \langle x' \rangle^{-k} \;\forall\alpha' \in \mathbb{N}^{n-1}, \;\forall x' \in \mathbb{R}^{n-1}, \;\forall k \ge 0.$$

Definition 4.1.6. A function $\mathbb{R}^{n-1} \times (0,\varepsilon_0] \times \mathbb{R}^n \ni (x',\varepsilon,\xi) \to t(x',\varepsilon,\xi)$ is called a trace symbol of order μ, if there exists a decomposition $t(x',\varepsilon,\xi) = t'(x',\varepsilon,\xi) + t_\infty(\varepsilon,\xi)$, such that

(i) the function t_∞ satisfies the conditions in Definition 3.1.26.

(ii) the function $\mathbb{R}^{n-1} \ni x' \to t'(x',\varepsilon,\xi)$ is smooth for $\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi \in \mathbb{R}^n$, and for $\forall\alpha' \in \mathbb{N}^{n-1}$, $x' \in \mathbb{R}^{n-1}$, the symbol $D^{\alpha'}_{x'} t'(x',\varepsilon,\xi)$ satisfies (3.1.26), where $C = C(\alpha',x')$ satisfies the inequality

$$|C(\alpha',x')| \le C_{k,\alpha'} \langle x' \rangle^{-k} \;\forall\alpha' \in \mathbb{N}^{n-1}, \;\forall x' \in \mathbb{R}^{n-1}. \;\forall k \ge 0.$$

We define the Poisson (trace) operator with the symbol k(t) by the formulae:

$$\pi_+ \operatorname{Op} k\, \psi = F^{-1}_{\xi\to x}\, k(x',\varepsilon,\xi)\, F_{x'\to\xi'}\psi$$

$$\pi_0 \operatorname{Op} t\, l_0 u = \pi_0\, F^{-1}_{\xi\to x}\, t(x',\varepsilon,\xi)\, F_{x\to\xi}\, l_0 u$$

The singular Green symbols depending upon x' and the corresponding operators are defined as in section 3.1. The boundedness of Poisson (trace, singular Green) operators can be stated and proved in an analogous way.

Let now p be a pseudodifferential symbol, g a singular Green symbol, t_i, $1 \le i \le s$, trace symbols, k_j, $1 \le j \le r$, Poisson symbols, and q_{ij}, $1 \le i \le s$, $1 \le j \le r$, pseudodifferential symbols on $\mathbb{R}^{n-1}$.

<u>Definition 4.1.7</u>. The matrix

$$\text{(4.1.7)} \qquad R = \begin{pmatrix} p(x,\varepsilon,\xi) + g(x',\varepsilon,\xi',\xi_n,\eta_n) & k_1(x',\varepsilon,\xi),\dots, k_r(x',\varepsilon,\xi) \\ t_1(x',\varepsilon,\xi) & q_{11}(x',\varepsilon,\xi'),\dots,q_{1r}(x',\varepsilon,\xi') \\ \vdots & \vdots \qquad\qquad \vdots \\ t_s(x',\varepsilon,\xi) & q_{s1}(x',\varepsilon,\xi'),\dots,q_{sr}(x',\varepsilon,\xi') \end{pmatrix}$$

is called singularly perturbed Wiener-Hopf symbol and the corresponding operator singularly perturbed Wiener-Hopf operator.

Let $t \in L_\mu$ be a trace symbol and let

$$\tau = s - \mu - \tfrac{1}{2}e_2$$

$$\sigma = \tau + (s_2-\mu_2-\tfrac{1}{2})e.$$

We say that the reduced operator of $\pi_0 \mathrm{Op}\, t \in \mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\tau)}(\mathbb{R}^{n-1}))$ is zero if $t \in L_{\mu-e}$, and that in this case, the order of approximation is $\gamma_0 = 1$. Moreover, the reduced operator of $\pi_0 \mathrm{Op}\, t \in \mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is said to be zero if $\mu_2+\tfrac{1}{2}-s_2 > 0$, and the order of approximation is the number $\gamma_0 = \min(1,\mu_2+\tfrac{1}{2}-s_2)$.

Let k be a Poisson symbol of order α and let

$$\tau = s + \alpha + \tfrac{1}{2}e_2$$

$$\sigma = \tau + (s_2+\alpha_2+\tfrac{1}{2})e.$$

The reduced operator of $\pi_+ \mathrm{Op}\, k \in \mathrm{Hom}(H_{(\tau)}(\mathbb{R}^{n-1}) H_{(s)}(\mathbb{R}^n_+))$ is said to be zero if k is a Poisson symbol of order α-e. In this case, the order of approximation is defined as $\gamma_0 = 1$. Moreover, we say that the reduced operator of $\pi_+ \mathrm{Op}\, k \in \mathrm{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(s)}(\mathbb{R}^n_+))$ is zero if $s_2+\alpha_2+\tfrac{1}{2} > 0$ and that the order of approximation is $\gamma_0 = \min(1,s_2+\alpha_2+\tfrac{1}{2})$.
For a Wiener-Hopf operator Op R, the order of approximation γ_0 is defined as the minimum of the orders of approximation of the elements of the matrix Op R.
For $U \subset \mathbb{R}^n$ a bounded domain with C^∞-boundary ∂U, we are going to introduce singularly perturbed Wiener-Hopf operators on U. Let the covering $(U_l)_{0\le l\le r}$, the diffeomorphisms $(\chi_l)_{1\le l\le r}$ and the partition of unity $(\psi_l)_{0\le l\le r}$ be as in section 2. Moreover, let $\Psi_l \in C_0^\infty(U_l)$ be functions such that $\Psi_l(x) \equiv 1 \;\forall x \in \mathrm{supp}\, \psi_l$.

With ξ, ξ_n the cotangential and conormal variables to ∂U, respectively, and with $x \in U$, $x' \in \partial U$, the matrix (4.1.7) is called singularly perturbed Wiener-Hopf symbol. Let $p_{(l)} + g_{(l)}$, $k_{j(l)}$, $t_{i(l)}$, $q_{ij(l)}$ denote the symbols $p+g$, k_j, t_i, q_{ij} in the variables $y \in \chi_l(U_l \cap U) \subset \mathbb{R}^n_+$. We define the singularly perturbed Wiener-Hopf operator with the symbol R by

$$\mathrm{Op}\ R = \begin{pmatrix} \pi\ \mathrm{Op}(p+g)\, l_0 & \pi\ \mathrm{Op}\ k_1 & \cdots & \mathrm{Op}\ k_r \\ \pi_0\ \mathrm{Op}\ t_1\ l_0 & \mathrm{Op}\ q_{11} & \cdots & \mathrm{Op}\ q_{1r} \\ & & & \\ \pi_0\ \mathrm{Op}\ t_s\ l_0 & \mathrm{Op}\ q_{s1} & \cdots & \mathrm{Op}\ q_{sr} \end{pmatrix},$$

where

$$\pi\ \mathrm{Op}(p+g)\, l_0 u = \pi \sum_{1 \le l \le r} \Psi_l(x)\, \mathrm{Op}(p_{(l)} + g_{(l)})(l_0 \psi_l u) + \pi \Psi_0(x)\, \mathrm{Op}\ p(l_0 \psi_0 u)$$

$$\pi\ \mathrm{Op}\ k_j \phi = \pi \sum_{1 \le l \le r} \Psi_l(x)\ \mathrm{Op}\ k_{j(l)}(\psi_l \phi)$$

$$\pi_0\ \mathrm{Op}\ t_i l_0 u = \pi_0 \sum_{1 \le l \le r} \Psi_l(x)\ \mathrm{Op}\ t_{i(l)}(l_0 \psi_l u)$$

$$\mathrm{Op}\ q_{ij}\ \phi = \sum_{1 \le l \le r} \Psi_l(x)\ \mathrm{Op}\ q_{ij(l)}\ (\psi_l \phi)$$

The claim of Lemma 3.1.31 holds, where X and Op X denote the sets of singularly perturbed Wiener-Hopf symbols and of the corresponding operators, respectively.

4.2. Statement of the main results

In this section, the results from Section 3.2 (two-sided a priori estimate for Wiener-Hopf operators on a half line) and Section 3.3 (Reduction to regular perturbations) are stated in the case of Wiener-Hopf operators on bounded domains (see [Fr 1] and [Fr-W] for the stability result and the construction of the reducing operator in the differential case).

Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂U. Let $Q^{(j)}(x,\varepsilon,\xi) \in P_{\nu^{(j)}}$, $j = 1,2$, be properly elliptic with $\nu_2^{(j)} = 2r_2^{(j)}$, $\nu_3^{(j)} = 2r_3^{(j)}$ and with

$$|Q^{(2)}(x,\varepsilon,\xi)| \geq C > 0 \quad \forall x \in \overline{U},\ \forall\varepsilon \in (0,\varepsilon_0],\ \forall\xi \in \mathbb{R}^n.$$

The rational function

$$(4.2.1) \qquad L(x,\varepsilon,\xi) = Q^{(1)}(x,\varepsilon,\xi)(Q^{(2)}(x,\varepsilon,\xi))^{-1}$$

is elliptic of order $\nu = \nu^{(1)}-\nu^{(2)}$ with index $r = (0,r_2^{(1)}-r_2^{(2)},r_3^{(1)}-r_3^{(2)})$. It is assumed that $r_2 \geq 0$, $r_3 \geq 0$.

Denote by λN the inward conormal and by ξ' the cotangential variables to ∂U at a point $x' \in \partial U$. With L_0 the principal symbol of L, let $L_0^+(x',\varepsilon,\xi'+\lambda N)$ be the factor in the symbol $\lambda \to L_0(x',\varepsilon,\xi'+\lambda N)$ which corresponds to the zeroes and singularities located in the half plane $\operatorname{Im}\lambda > 0$ for $x' \in \partial U$, $\varepsilon \in (0,\varepsilon_0]$, $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$. According to lemma 3.1.10, L_0^+ can be factorized as follows:

$$(4.2.2) \qquad L_0^+(x',\varepsilon,\xi'+\lambda N) = L_{01}^+(x',\varepsilon,\xi'+\lambda N)\ L_{02}^+(x',\varepsilon,\xi'+\lambda N)$$

where L_{01}^+, L_{02}^+ are homogeneous in $(\varepsilon^{-1},\xi',\lambda)$ of order r_2 and 0, respectively, and where the inequalities

$$C^{-1}(|\xi'|+|\lambda|)^{r_2} \leq |L_{01}^+(x',\varepsilon,\xi'+\lambda N)| \leq C(|\xi'|+|\lambda|)^{r_2}$$

$$C^{-1}(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3} \le |L_{02}^{+}(x',\varepsilon,\xi'+\lambda N)| \le C(1+\varepsilon|\xi'|+\varepsilon|\lambda|)^{r_3}$$

hold for $\forall\varepsilon \in (0,\varepsilon_0]$, $\forall\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $\forall\lambda \in \mathbb{C}$, $\mathrm{Im}\,\lambda < 0$, $\forall x' \in \partial U$ with some constant C. Let the symbols L_0^{0+}, L^+ and L_{00}^+ be defined by

$$(4.2.3) \qquad L_0^{0+}(x',\xi'+\lambda N) = \lim_{\varepsilon\downarrow 0} L_{01}^{+}(x',\varepsilon,\xi'+\lambda N)$$

$$(4.2.4) \qquad L^{+}(x',\lambda) = L_{02}^{+}(x',1,\lambda N)$$

$$(4.2.5) \qquad L_{00}^{+}(x',\xi'+\lambda N) = \lim_{\rho\to\infty} \rho^{-r_3} L_0^{+}(x',\rho,\xi'+\lambda N).$$

Let $\Gamma_1,\Gamma_2,\Gamma_3$ be closed curves in the half plane $\{\mathrm{Im}\,\lambda > 0\}$, which enclose all the zeroes of the rational functions $\lambda \to L_0^{0+}(x',\omega'+\lambda N)$, $\lambda \to L^{+}(x',\lambda)$, $\lambda \to L_{00}^{+}(x',\omega'+\lambda N)$, respectively, for $\forall x' \in \partial U$, $\forall\omega' \in \Omega_{n-1}$.
Let $L_j(x',\varepsilon,\xi)$, $1 \le j \le l_1$, be trace symbols with principal parts L_{j0} and with orders $\mu_j = (\gamma_j,m_j,p_j)$. Without loss of generality, it is assumed that L_j are ordered in such a way that $m_1 \le m_2 \le \ldots \le m_{l_1}$.
With π and π_0 the restriction operators to U and ∂U, respectively, consider the following singularly perturbed boundary value problem:

$$(4.2.6) \qquad \pi L(x,\varepsilon,-i\tfrac{\partial}{\partial x})u = f(x), \qquad x \in U$$

$$(4.2.7) \qquad \pi_0 L_j(x',\varepsilon,-i\tfrac{\partial}{\partial x})u = \phi_j(x'), \quad x' \in \partial U,\ 1 \le j \le l_1 ,$$

where f, ϕ_j are given and where the support of the solution u is contained in $\overline{U}$. If $l_1 = r_2+r_3$ and if the numbers m_j, r_2, k_0 satisfy the condition

$$(4.2.8) \qquad \max(r_2-1,m_{r_2}) < \min(k_0,m_{r_2+1}),$$

then let l_0, α, β be defined by

$$(4.2.9) \quad \begin{cases} l_0 = r_2 \\ \alpha = \max(r_2-1,m_{r_2}) \\ \beta = \min(k_0,m_{r_2+1}). \end{cases}$$

The solution of (4.2.6), (4.2.7) is sought in the space $H_{(s)}(U)$, where $s \in \mathbb{R}^3$ satisfies the conditions

$$(4.2.10) \quad \alpha+\tfrac{1}{2} < s_2 < \beta+\tfrac{1}{2}$$

$$(4.2.11) \begin{cases} s_2+s_3 > \max(\max\limits_{1\le j\le r_2+r_3}(m_j+p_j), r_2+r_3-1)+\tfrac{1}{2} \\ s_3 > 0. \end{cases}$$

It is assumed that the right hand side f of (4.2.6) belongs to the space $H_{(s-\nu)}(U)$.

The principal parts $L_{j0}(x',\varepsilon,\xi'+\lambda N)$ of the symbols $L_j(x',\varepsilon,\xi'+\lambda N)$ are supposed to satisfy the following <u>coerciveness condition</u>, <u>introduced in [Fr 1]</u>:

(i) With $L_{j0}^{0}(x',\xi) = \lim\limits_{\varepsilon\to 0} \varepsilon^{\gamma_j} L_{j0}(x',\varepsilon,\xi)$ the reduced symbol for $L_{j0}(x',\varepsilon,\xi)$ and with Γ_1 the curve defined above, introduce

$$(4.2.12) \quad q_{kj}^{0}(x',\omega') = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{L_{j0}^{0}(x',\omega'+\lambda N)\lambda^{k-1}}{L_0^{0+}(x',\omega'+\lambda N)}\, d\lambda, \quad 1 \le k,j \le l_0.$$

The matrix $||q_{kj}^{0}(x',\omega')||_{1\le k,j\le l_0}$ is supposed to be non-singular:

$$(4.2.13) \quad \det||q_{kj}^{0}(x',\omega')||_{1\le k,j\le l_0} \neq 0, \ \forall x' \in \partial U, \ \forall \omega' \in \Omega_{n-1}.$$

(ii) With $L_j(x',\lambda) = L_{j0}(x',1,\lambda N)$ and with Γ_2 the curve defined above, introduce

$$(4.2.14) \quad q_{kj}(x') = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{L_j(x',\lambda)\lambda^{k-r_2-1}}{L^{+}(x',\lambda)}\, d\lambda, \quad l_0 \le k,\ j \le l_1$$

The matrix $||q_{kj}(x')||$ is supposed to be non-singular:

$$(4.2.15) \quad \det||q_{kj}(x')||_{l_0\le k,j\le l_1} \neq 0, \quad \forall x' \in \partial U.$$

(iii) With $L_{j00}(x',\omega'+\lambda N) = \lim\limits_{\rho\to\infty} \rho^{\gamma_j-p_j} L_{j0}(x',\rho,\omega'+\lambda N)$ and with Γ_3 the curve defined above, introduce

$$(4.2.16)\quad q_{kj}^{00}(x',\omega') = \frac{1}{2\pi i} \int_{\Gamma_3} \frac{L_{j00}(x',\omega'+\lambda N)}{L_{00}^{+}(x',\omega'+\lambda N)} \lambda^{k-1} d\lambda,\ 1 \le k,\ j \le l_1.$$

The matrix $||q_{kj}^{00}(x',\omega')||$ is supposed to be non-singular:

$$(4.2.17)\quad \det||q_{kj}^{00}(x',\omega')||_{1\le k,j\le l_1} \ne 0,\ \forall x' \in \partial U,\ \forall \omega' \in \Omega_{n-1}.$$

(iv) Introduce

$$(4.2.18)\quad Q_{kj}(x',\rho,\omega') = \frac{1}{2\pi i}\int_{\gamma(x',\rho,\omega')} \rho^{\gamma_j - p_j + r_2} \frac{L_{j0}(x',\rho,\omega'+\lambda N)}{L_0^{+}(x',\rho,\omega'+\lambda N)} \lambda^{k-1} d\lambda,\ 1 \le k,\ j \le l_1$$

where $\gamma(x',\rho,\omega')$ is a closed curve in the half plane $\{\mathrm{Im}\,\lambda > 0\}$ which encloses all the zeroes of $\lambda \to L_0^{+}(x',\rho,\omega'+\lambda N)$. The matrix $||Q_{kj}||$ is supposed to be non-singular:

$$(4.2.19)\quad \det||Q_{kj}(x',\rho,\omega')||_{1\le k,j\le l_1} \ne 0,\ \forall (x',\rho,\kappa') \in \partial U \times (0,\infty) \times \Omega_{n-1}.$$

The stability result for singularly perturbed Wiener-Hopf operators on the half line (Theorems 3.2.1, 3.2.9) will be carried over to Wiener-Hopf operators in a bounded domain, using the partition of unity technique (see Theorem 3.2.5 in [Fr 1], where the two-sided a priori estimate for coercive differential singular perturbations in bounded domains was established).

Theorem 4.2.1.

Assume that $l_1 = r_2+r_3$, that (4.2.8) holds, that the symbol $\xi \to L(x,\varepsilon,\xi)$ is elliptic of order ν, and that the symbols $L_j(x',\varepsilon,\xi)$, $1 \le j \le r_2+r_3$, satisfy the coerciveness condition (i)-(iv). Then the following two-sided a priori estimate holds for solutions of (4.2.6), (4.2.7):

$$(4.2.20)\quad C^{-1}(||f||_{(s-\nu)} + \sum_{1\le j\le r_2} [\phi_j]_{(\tau_j)} + \sum_{r_2<j\le r_2+r_3} [\phi_j]_{(\sigma_j)} + ||u||_{(s-e_2)}) \le$$

$$\le ||u||_{(s)} \le C(||f||_{(s-\nu)} + \sum_{1\le j\le r_2} [\phi_j]_{(\tau_j)} + \sum_{r_2<j\le r_2+r_3} [\phi_j]_{(\sigma_j)} +$$

$$+||u||_{(s-e_2)})$$

where $s \in \mathbb{R}^3$ satisfies (4.2.10), (4.2.11), C is independent upon $u, f, \phi_j, \varepsilon$, and where

$$\tau_j = s-\mu_j-\tfrac{1}{2}e_2$$
$$\sigma_j = \tau_j+(s_2-m_2-\tfrac{1}{2})e.$$

Now we return to the problem (4.2.6), (4.2.7) and drop the assumption $l_1 = r_2+r_3$. Instead, we assume that there exists s_2+s_3 which satisfies the conditions

(4.2.21) $\quad r_2+r_3-l_1-\tfrac{1}{2} < s_2+s_3 < r_2+r_3-l_1+\tfrac{1}{2}$

(4.2.22) $\quad \max\limits_{1\le j\le l_1} (m_j+p_j)+\tfrac{1}{2} < s_2+s_3.$

If there exists l, $0 \le l \le l_1$, such that with $m_0 = -\infty$, $m_{l_1+1} = \infty$, one has:

(4.2.23) $\quad m_l \le r_2-l-1 < m_{l+1}$,

then l is unique and we introduce l_0, α, β by

$$(4.2.24)\begin{cases} l_0 = l \\ \alpha = r_2-l_0-1 \\ \beta = r_2-l_0. \end{cases}$$

The solution of (4.2.6), (4.2.7) is sought in the space $\mathring{H}_{(s)}(U)$, where $s \in \mathbb{R}^3$ satisfies (4.2.21), (4.2.22) and

(4.2.25) $\quad \alpha+\tfrac{1}{2} < s_2 < \beta+\tfrac{1}{2}.$

It is assumed that the right hand side f of (4.2.6) belongs to the space $H_{(s-\nu)}(U)$. One has the

Theorem 4.2.2. Assume that the rational symbol $\xi \to L(x,\varepsilon,\xi)$ is elliptic of order ν and that the symbols $L_j(x',\varepsilon,\xi)$, $1 \le j \le l_1$, satisfy the coerciveness condition (i)-(iv) with l_0 determined by (4.2.23). Then the following two-sided a priori estimate holds for solutions of (4.2.6), (4.2.7):

$$(4.2.26)\quad C^{-1}(||f||_{(s-\nu)}+\sum_{1\le j\le l_0}[\phi_j]_{(\tau_j)}+\sum_{l_0<j\le l_1}[\phi_j]_{(\sigma_j)}+||u||_{(s-e_2)})\le$$

$$\le ||u||_{(s)}\le C(||f||_{(s-\nu)}+\sum_{1\le j\le l_0}[\phi_j]_{(\tau_j)}+\sum_{l_0<j\le l_1}[\phi_j]_{(\sigma_j)}+||u||_{(s-e_2)})$$

where $s \in \mathbb{R}^3$ satisfies (4.2.21), (4.2.25), C is independent upon u,f,ϕ_j and where τ_j,σ_j are as in Theorem 4.2.1.

The Theorems 4.2.1, 4.2.2 will be proved in section 4.3 below. Now we are going to construct the reducing operator for coercive singularly perturbed Wiener-Hopf operators in a bounded domain (see [Fr-W] where this construction was given in the differential case). We associate with (4.2.6), (4.2.7) the Wiener-Hopf symbols

$$A^\varepsilon=\begin{pmatrix} L(x,\varepsilon,\xi)\\ L_1(x',\varepsilon,\xi)\\ \vdots\\ L_{l_1}(x',\varepsilon,\xi)\end{pmatrix},\quad A_0^\varepsilon=\begin{pmatrix} L_0(x,\varepsilon,\xi)\\ L_{10}(x',\varepsilon,\xi)\\ \vdots\\ L_{l_10}(x',\varepsilon,\xi)\end{pmatrix},$$

$$A^0=\begin{pmatrix} L_0^0(x,\xi)\\ L_1^0(x',\xi)\\ \vdots\\ L_{l_0}^0(x',\xi)\end{pmatrix},\quad A_0^0=\begin{pmatrix} L_0^0(x,\xi)\\ L_{10}^0(x',\xi)\\ \vdots\\ L_{l_00}^0(x',\xi)\end{pmatrix}.$$

Let the covering $(U_l)_{0\le l\le r}$ of U, the diffeomorphisms χ_l, $1\le l\le r$, and the partition of unity $(\phi_l)_{0\le l\le r}$ be as in Section 2.1. Moreover, let $\Psi_l \in C_0^\infty(\mathbb{R}^n)$, $0\le l\le r$, be functions such that supp $\Psi_l\subset U_l$ and $\Psi_l(x)\equiv 1$, $\forall x \in$ supp ψ_l. With $y=(y',y_n)\in\chi_l(U_l\cap\overline{U})$ and η the dual variable to y, denote by $A^\varepsilon_{(l)0}$ and $A^0_{(l)0}$ the symbols A_0^ε and A_0^0 in the new coordinates (y,η):

$$A^{\varepsilon}_{(1)0} = \begin{pmatrix} L_{(1)0}(y,\varepsilon,\eta) \\ L_{(1)10}(y',\varepsilon,\eta) \\ \vdots \\ L_{(1)1_10}(y',\varepsilon,\eta) \end{pmatrix}, \quad A^{0}_{(1)0} = \begin{pmatrix} L^0_{(1)0}(y,\eta) \\ L^0_{(1)10}(y',\eta) \\ \vdots \\ L_{(1)1_00}(y',\eta) \end{pmatrix}.$$

Denote by π_0 and π_+ the restriction operators to the point $y_n = 0$ and the half line $\mathbb{R}_+$, respectively, and by l_0 the extension operator by zero to $\mathbb{R}$. Further, let $\widehat{\eta'} = \langle\eta'\rangle|\eta'|^{-1}\eta'$. Since the reduced problem satisfies the coerciveness condition, the following boundary value problem on the half line $y_n > 0$ with $\eta' \neq 0$, y' as parameters has a unique solution $v_{(1)}$:

$$\pi_+ L^0_{(1)0}(y',0,\widehat{\eta'},-i\frac{\partial}{\partial y_n})v_{(1)}(y_n) = 0, \qquad y_n > 0$$

$$\pi_0 L^0_{(1)j0}(y',\widehat{\eta'},-i\frac{\partial}{\partial y_n})v_{(1)} = \hat{\phi}_j - \pi_0 L^0_{(1)j0}(y',\widehat{\eta'},-i\frac{\partial}{\partial y_n})w_{(1)}, \quad 1 \le j \le l_0$$

$$\lim_{y_n \to \infty} v_{(1)}(y_n) = 0,$$

where $w_{(1)}$ is given by

$$w_{(1)} = \pi_+ (L_0^{0+})^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})\, l_0 \pi_+ (L_0^{0-})^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})\, lf.$$

Therefore, an operator $\mathrm{Op}\,\widehat{C^0_{(1)0}}$ can be defined as follows:

$$((\mathrm{Op}\,\widehat{C^0_{(1)0}})((f,\phi_1,\ldots,\phi_{1_1})^T)(y',y_n) =$$
$$= F^{-1}_{\eta'\to y'}(v_{(1)}(y_n)+w_{(1)}(y_n))$$

where $\widehat{C^0_{(1)0}}$ is the inverse symbol of $\widehat{A^0_{(1)0}}$ which is obtained from $A^0_{(1)0}$ by replacing η' with $\widehat{\eta'}$. The coerciveness condition allows us to construct the solution $v_{(1)}$ of the following singularly perturbed boundary value problem on the half line $y_n > 0$ with η',ε,y' as parameters:

$$\pi_+ L_{(1)0}(y',0,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})v_{(1)}(y_n) = 0, \quad y_n > 0$$

$$\pi_0 L_{(1)j0}(y',\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})v_{(1)} =$$

$$= \phi_j - \pi_0 \operatorname{Op} L_{(1)j0}(y',\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})w_{(1)}, \quad 1 \le j \le l_1$$

$$\lim_{y_n \to \infty} v_{(1)}(y_n) = 0$$

where $w_{(1)}$ is constructed as in section 3.2, namely

$$w = \pi_+ (L_{01}^+ d)^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n}) l_0 \pi_+ (L_0^- c)^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n}) lf$$

if the assumptions of Theorem 4.2.1 are satisfied, and

$$w = (L_0^+)^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})(\frac{\partial}{\partial y_n}+\langle\eta'\rangle)^{l_0}(\varepsilon\frac{\partial}{\partial y_n}+\langle\varepsilon\eta'\rangle)^{l_1-l_0} l_0 \pi_+$$

$$(L_0^-)^{-1}(y,\varepsilon,\widehat{\eta'},-i\frac{\partial}{\partial y_n})(\frac{\partial}{\partial y_n}+\langle\eta'\rangle)^{-l_0}(\varepsilon\frac{\partial}{\partial y_n}+\langle\varepsilon\eta'\rangle)^{l_0-l_1} lf$$

if the assumptions of Theorem 4.2.2 are satisfied. Therefore, one can define the operator $\operatorname{Op} \widehat{C^\varepsilon_{(1)0}}$ as follows:

$$((\operatorname{Op} \widehat{C^\varepsilon_{(1)0}}((f,\phi_1,\ldots,\phi_{l_1})^T)(y',y_n) =$$

$$= F^{-1}_{\eta'\to y'}(v_{(1)}(y_n)+w_{(1)}(y_n)).$$

$\widehat{C^\varepsilon_{(1)0}}$ is the inverse symbol of $\widehat{A^\varepsilon_{(1)0}}$ which is obtained by replacing in $A^\varepsilon_{(1)0}$ the variable η' with $\widehat{\eta'}$.

Under the conditions of Theorem 4.2.1, we define the symbols p_1, p_2 and the operators R^ε_r, R^ε_l, S^ε_r, S^ε_l by:

(4.2.27) $\quad p_1(\varepsilon,\xi) \equiv 1, \; p_2(\varepsilon,\xi) \equiv 1$

(4.2.28) $\quad R^\varepsilon_l((f,\phi_1,\ldots,\phi_{l_0})^T)(x) =$

$$= \sum_{l=1}^{r} \Psi_l(x) \operatorname{Op}(\widehat{A^\varepsilon_{(1)0}} \circ p_1^{-1} \circ \widehat{C^0_{(1)0}})(\psi_l f, \psi_l\phi_1, \ldots, \psi_l\phi_{l_0})^T +$$

$$+ (\Psi_0(x) \operatorname{Op} L_0^\varepsilon (L_0^0)^{-1} p_2^{-1}(x,\varepsilon,\hat{\xi})(\psi_0 f), \underbrace{0,\ldots,0}_{l_1})^T$$

$$(4.2.29)\quad S_1^\varepsilon((f,\phi_1,\dots,\phi_{1_1})^T)(x) =$$
$$= \sum_{l=1}^{r} \Psi_l(x)\ \mathrm{Op}(\widehat{A^0_{(l)0}} \circ p_2 \circ \widehat{C^\varepsilon_{(l)0}})(\psi_l f,\psi_l\phi_1,\dots,\psi_l\phi_{1_1})^T +$$
$$+ (\Psi_0(x)\ \mathrm{Op}(L_0^0(L_0^\varepsilon)^{-1}p_2(x,\varepsilon,\hat{\xi})(\psi_0 f),\underbrace{0,\dots,0}_{1_0})$$

$$(4.2.30)\quad \mathcal{R}_r^\varepsilon = \mathrm{Id},\quad \mathcal{S}_r^\varepsilon = \mathrm{Id}.$$

Moreover, we introduce the following spaces:

$$\mathcal{H} \overset{\text{def}}{=} H_{(s)}(U)$$
$$\mathcal{K} \overset{\text{def}}{=} H_{(s-\nu)}(U) \times \prod_{1\le j\le r_2} H_{(\tau_j)}(\partial U) \times \prod_{r_2<j\le r_2+r_3} H_{(\sigma_j)}(\partial U)$$
$$\mathcal{H}^0 \overset{\text{def}}{=} H_{(s)}(U)$$
$$\mathcal{K}^0 \overset{\text{def}}{=} H_{(s-\nu_2 e_2)}(U) \times \prod_{1\le j\le r_2} H_{(s-(m_j+\frac12)e_2)}(\partial U).$$

Assume now that the conditions of Theorem 4.2.2 are satisfied. Then define the symbols p_1,p_2 and the operators $\mathcal{R}_r^\varepsilon$, $\mathcal{S}_r^\varepsilon$ by

$$(4.2.31)\quad p_1(\varepsilon,\xi) = \varepsilon^{-s_1}(i\varepsilon\xi_N+\langle\varepsilon\xi'\rangle)^{s_3},\quad p_2(\varepsilon,\xi) = \varepsilon^{-s_1}\langle\varepsilon\xi\rangle^{s_3}$$
$$(4.2.32)\quad (\mathcal{R}_r^\varepsilon u)(x) = \sum_{l=1}^{r} \Psi_l(x)\ \mathrm{Op}\ p_1(\psi_l u) + \Psi_0(x)\ \mathrm{Op}\ p_2(\psi_0 u)$$
$$(4.2.33)\quad (\mathcal{S}_r^\varepsilon u)(x) = \sum_{l=1}^{r} \Psi_l(x)\ \mathrm{Op}\ p_1^{-1}(\psi_l u) + \Psi_0(x)\ \mathrm{Op}\ p_2^{-1}(\psi_0 u).$$

The operators $\mathcal{R}_1^\varepsilon$, $\mathcal{S}_1^\varepsilon$ are defined by (4.2.28), (4.2.29) with p_1, p_2 as in (4.2.31). Moreover, we introduce the spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{H}^0$, $\mathcal{K}^0$ as follows:

$$\mathcal{H} \overset{\text{def}}{=} \mathring{H}_{(s)}(U)$$
$$\mathcal{K} \overset{\text{def}}{=} H_{(s-\nu)}(U) \times \prod_{1\le j\le 1_0} H_{(\tau_j)}(\partial U) \times \prod_{1_0<j\le 1_1} H_{(\sigma_j)}(\partial U)$$
$$\mathcal{H}^0 \overset{\text{def}}{=} \mathring{H}_{s_2}(U)$$
$$\mathcal{K}^0 \overset{\text{def}}{=} H_{s_2-\nu_2}(U) \times \prod_{1\le j\le 1_0} H_{s_2-m_j-\frac12}(\partial U).$$

Let $s \in \mathbb{R}^3$ be such that the conditions in section 3.2 are satisfied and let $\gamma_0 = \min(1, \beta+\frac{1}{2}-s_2)$. Moreover, let

$$(4.2.34) \qquad \mathcal{A}^\varepsilon \overset{\text{def}}{=} \text{Op } A^\varepsilon, \quad \mathcal{A}^0 \overset{\text{def}}{=} \text{Op } A^0.$$

Then one has:

Theorem 4.2.3. Let $\mathcal{A}^\varepsilon$ be a coercive singularly perturbed Wiener-Hopf operator in the domain U. With R_r^ε, R_1^ε, S_r^ε, S_1^ε defined above, the diagram

$$(4.2.35) \qquad \begin{tikzcd} H \arrow[r, "\mathcal{A}^\varepsilon"] \arrow[d, shift left, "R_r^\varepsilon"] & K \arrow[d, shift right, "S_1^\varepsilon"'] \\ H^0 \arrow[r, "\mathcal{A}^0"] \arrow[u, shift left, "S_r^\varepsilon"] & K^0 \arrow[u, shift right, "R_1^\varepsilon"'] \end{tikzcd}$$

is commutative modulo operators of a norm bounded by $C\,\varepsilon^{\gamma_0}|\ln \varepsilon|$ with some constant $C > 0$. In other words, the following inequalities hold for $\varepsilon \in (0,\varepsilon_0]$:

$$(4.2.36) \qquad ||\mathcal{A}^\varepsilon - R_1^\varepsilon \mathcal{A}^0 R_r^\varepsilon||_{\text{Hom}(H,K)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon|$$

$$(4.2.37) \qquad \begin{cases} ||R_r^\varepsilon S_r^\varepsilon - \text{Id}||_{\text{Hom}(H^0,H^0)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \\ ||R_1^\varepsilon S_1^\varepsilon - \text{Id}||_{\text{Hom}(K,K)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \end{cases}$$

$$(4.2.38) \qquad \begin{cases} ||S_r^\varepsilon R_r^\varepsilon - \text{Id}||_{\text{Hom}(H,H)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \\ ||S_1^\varepsilon R_1^\varepsilon - \text{Id}||_{\text{Hom}(K^0,K^0)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon| \end{cases}$$

$$(4.2.39) \qquad ||\mathcal{A}^0 - S_1^\varepsilon \mathcal{A}^\varepsilon S_r^\varepsilon||_{\text{Hom}(H^0,K^0)} \le C\,\varepsilon^{\gamma_0}|\ln \varepsilon|$$

where the constant C does not depend upon ε.

Remark 4.2.4. In the special case of the linear plate problem, Theorem 4.2.3 was proved in [Fr-W] Moreover, in [Fr-W], the commutativity of

(4.2.35) was stated without proof for singularly perturbed differential boundary value problems.

Corollary 4.2.5. Assume that the inverse operator $(\mathfrak{A}^0)^{-1} \in \mathrm{Hom}(K^0, H^0)$ of $\mathfrak{A}^0$ exists. Then the inverse operator $(\mathfrak{A}^\varepsilon)^{-1} \in \mathrm{Hom}(K,H)$ exists for $\forall\varepsilon \in (0,\varepsilon_0]$ with ε_0 sufficiently small, and it can be expanded in a convergent series as follows:

$$(4.2.40) \quad (\mathfrak{A}^\varepsilon)^{-1} = S_r^\varepsilon \sum_{k\geq 0} [(\mathfrak{A}^0)^{-1}(\mathfrak{A}^0 - S_l^\varepsilon \mathfrak{A}^\varepsilon S_r^\varepsilon)]^k (\mathfrak{A}^0)^{-1} S_l^\varepsilon.$$

The following result states the stability of the index

$$\kappa(\varepsilon) \overset{\text{def}}{=} \dim(\ker \mathfrak{A}^\varepsilon) - \dim(\operatorname{coker} \mathfrak{A}^\varepsilon)$$

with respect to coercive singular perturbations:

Corollary 4.2.6. $\kappa(\varepsilon) = \kappa(0), \quad \forall\varepsilon > 0.$

4.3. Proof of the Theorems 4.2.1, 4.2.2, 4.2.3

In order to prove the theorems stated in section 4.2, we need several auxiliary results.

The following lemma, being a slight generalisation of Lemma 2.4.7 in [Fr 1], will be proved using a technique from [K-N].

Lemma 4.3.1. Let $L_j(x,\varepsilon,\xi) \in \mathcal{L}_\nu(j)$, $j = 1,2$. Then the operator

$$K = \mathrm{Op}\, L_1(x,\varepsilon,\xi) \circ \mathrm{Op}\, L_2(x,\varepsilon,\xi) - \mathrm{Op}(L_1L_2)(x,\varepsilon,\xi) \tag{4.3.1}$$

has order $\mu = \nu^{(1)}+\nu^{(2)}-e_2$.

Proof. The symbols L_j can be decomposed as follows:

$$L_j(x,\varepsilon,\xi) = L_j'(x,\varepsilon,\xi)+L_{j\infty}(\varepsilon,\xi),$$

where $L_{j\infty}$ does not depend upon x and where $L_j'(x,\varepsilon,\xi)$ is rapidly decreasing as $|x| \to \infty$. Since the symbol $L_{2\infty}$ does not depend upon x, one has:

$$\mathrm{Op}\, L_1(x,\varepsilon,\xi) \circ \mathrm{Op}\, L_{2\infty}(\varepsilon,\xi) - \mathrm{Op}(L_1L_{2\infty})(x,\varepsilon,\xi) = 0.$$

Thus, it suffices to prove that

$$\mathrm{Op}\, L_1'(x,\varepsilon,\xi) \circ \mathrm{Op}\, L_2'(x,\varepsilon,\xi) - \mathrm{Op}(L_1'L_2')(x,\varepsilon,\xi) = R' \tag{4.3.2}$$

$$\mathrm{Op}\, L_{1\infty}(\varepsilon,\xi) \circ \mathrm{Op}\, L_2'(x,\varepsilon,\xi) - \mathrm{Op}(L_{1\infty}L_2')(x,\varepsilon,\xi) = R_\infty, \tag{4.3.3}$$

where R' and R_∞ have both at most order μ.

First, we prove (4.3.2). One has:

$$(R'u)\hat{}\,(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} K'(\xi,\varepsilon,\eta)\hat{u}(\eta)d\eta$$

where K' is given by

$$K'(\xi,\varepsilon,\eta) = \int_{\mathbb{R}^n} (\widehat{L_1'}(\xi-\tau,\varepsilon,\tau)-\widehat{L_1'}(\xi-\tau,\varepsilon,\eta))\widehat{L_2'}(\tau-\eta,\varepsilon,\eta)d\tau \tag{4.3.4}$$

and where the hat denotes the Fourier transform with respect to the first variable. In order to prove

$$(4.3.5) \qquad ||R'u||_{(s)} \le C_s ||u||_{(s+\mu)} \qquad \forall s \in \mathbb{R}^3,$$

by Schur's Lemma on the boundedness of integral operators in L^2 it is sufficient to show that

$$(4.3.5) \qquad \varepsilon^{\mu_1} \int \langle\xi\rangle^{s_2} \langle\varepsilon\xi\rangle^{s_3} \langle\eta\rangle^{-(s_2+\mu_2)} \langle\varepsilon\eta\rangle^{-(s_3+\mu_3)} |K'(\xi,\varepsilon,\eta)|\,d\xi \le C$$

$$(4.3.6) \qquad \varepsilon^{\mu_1} \int \langle\xi\rangle^{s_2} \langle\varepsilon\xi\rangle^{s_3} \langle\eta\rangle^{-(s_2+\mu_2)} \langle\varepsilon\eta\rangle^{-(s_3+\mu_3)} |K'(\xi,\varepsilon,\eta)|\,d\eta \le C$$

with some constant C which does not depend upon ε,η and ξ. The definition of L_ν yields:

$$|D_x^\alpha(L_1'(x,\varepsilon,\tau)-L_1'(x,\varepsilon,\eta))| \le C_{\alpha,k}\varepsilon^{-\nu_1^{(1)}} \langle\tau-\eta\rangle^{m_1} \langle\eta\rangle^{\nu_2^{(1)}-1} \langle\varepsilon\eta\rangle^{\nu_3^{(1)}} \langle x\rangle^{-k}$$

$$\forall\alpha \in \mathbb{N}^n, \ \forall x,\tau,\eta \in \mathbb{R}^n, \forall k > 0.$$

$$|D_x^\alpha L_2'(x,\varepsilon,\eta)| \le C_{\alpha,k}\varepsilon^{-\nu_1^{(2)}} \langle\eta\rangle^{\nu_2^{(2)}} \langle\varepsilon\eta\rangle^{\nu_3^{(2)}} \langle x\rangle^{-k}$$

$$\forall\alpha \in \mathbb{N}^n, \ \forall x,\eta \in \mathbb{R}^n, \ \forall k > 0$$

with constants $C_{\alpha,k}$ which depend only upon their subscripts. Hence,

$$|\widehat{L_1'}(\xi-\tau,\varepsilon,\tau)-\widehat{L_1'}(\xi-\tau,\varepsilon,\eta)| \le C_k\varepsilon^{-\nu_1^{(1)}} \langle\tau-\eta\rangle^{m_1} \langle\eta\rangle^{\nu_2^{(1)}-1} \langle\varepsilon\eta\rangle^{\nu_3^{(1)}} \langle\xi-\tau\rangle^{-k}$$

$$|\widehat{L_2'}(\tau-\eta,\varepsilon,\eta)| \le C_k\varepsilon^{-\nu_1^{(2)}} \langle\eta\rangle^{\nu_2^{(2)}} \langle\varepsilon\eta\rangle^{\nu_3^{(2)}} \langle\tau-\eta\rangle^{-k}.$$

Thus the function K' can be estimated as follows:

$$|K'(\xi,\varepsilon,\eta)| \le C_k\varepsilon^{-\mu_1} \langle\eta\rangle^{\mu_2} \langle\varepsilon\eta\rangle^{\mu_3} \int_{\mathbb{R}^n} \langle\eta-\tau\rangle^{m_1-k} \langle\xi-\tau\rangle^{-k}\,d\tau.$$

Using again the inequality (4.1.2), we can estimate the left hand side of (4.3.5) as follows:

$$\varepsilon^{\mu_1}\int\langle\xi\rangle^{s_2}\langle\varepsilon\xi\rangle^{s_3}\langle\eta\rangle^{-(s_2+\mu_2)}\langle\varepsilon\eta\rangle^{-(s_3+\mu_3)}|K'(\xi,\varepsilon,\eta)|d\xi \le$$

$$\le C_k\iint\langle\eta-\tau\rangle^{m_1-k}\langle\xi-\tau\rangle^{-k}\langle\xi-\eta\rangle^{|s_2|+|s_3|}d\xi d\tau$$

$$\le C_k\iint\langle\xi-\tau\rangle^{m_2-k}d\xi\langle\eta-\tau\rangle^{m_3-k}d\tau$$

with

$m_2 = |s_2|+|s_3|$

$m_3 = m_1+|s_2|+|s_3|$.

For k sufficiently large (namely, $k > n+\max(m_2,m_3)$), the last integral is bounded uniformly w.r.t. $\eta \in \mathbb{R}^n$. That ends the proof of (4.3.5). The same argument can be used in order to prove (4.3.6).

Using the inequality

$$|(L_{1\infty}(\varepsilon,\xi)-L_{1\infty}(\varepsilon,\eta))| \le C\varepsilon^{-\nu_1^{(1)}}\langle\xi-\eta\rangle^{m_1}\langle\eta\rangle^{\nu_2^{(1)}-1}\langle\varepsilon\eta\rangle^{\nu_3^{(1)}},$$

one gets (4.3.5), (4.3.6) with K' replaced by the function

$$K_\infty(\xi,\varepsilon,\eta) = (L_{1\infty}(\varepsilon,\xi)-L_{1\infty}(\varepsilon,\eta))\,\widehat{L_2'}(\xi-\eta,\varepsilon,\eta).$$

Therefore, the operator R_∞ defined in (4.3.3) has at most order μ. □

Lemma 4.3.2. Under the assumptions of Lemma 4.3.1, the commutator $[\mathrm{Op}\, L_1, \mathrm{Op}\, L_2] = \mathrm{Op}\, L_1 \circ \mathrm{Op}\, L_2 - \mathrm{Op}\, L_2 \circ \mathrm{Op}\, L_1$ has at most order μ.

Proof. Since $[\mathrm{Op}\, L_1, \mathrm{Op}\, L_2] = (\mathrm{Op}\, L_1 \circ \mathrm{Op}\, L_2 - \mathrm{Op}(L_1.L_2))+(\mathrm{Op}(L_1.L_2)- \mathrm{Op}\, L_2 \circ \mathrm{Op}\, L_1)$, this statement is a direct consequence of Lemma 4.3.1. □

Proposition 4.3.3. ([Fr 1]). Let $L(x,\varepsilon,\xi) \in \mathcal{L}_\nu$ be elliptic of order ν. Then for $\varepsilon \in (0,\varepsilon_0]$ with ε_0 sufficiently small, the a priori estimate holds:

$$(4.3.7)\qquad ||u||_{(s)} \le C_s(||L(x,\varepsilon,-i\tfrac{\partial}{\partial x})u||_{(s-\nu)}+||u||_{(s-e_2)})\ \ \forall u \in H_{(s)}(\mathbb{R}^n)$$

for $\forall s \in \mathbb{R}^3$, where the constant C_s does not depend on ε and u.

Proof. See the proof of Theorem 2.4.10 in [Fr 1]. $\square$

From now on, we are going to use the notation from section 4.2. Moreover, for $\forall x' \in \partial U$, $\forall \delta > 0$, we introduce $V_{x',\delta} = \{x \in U \mid |x-x'| < \delta\}$.

Proposition 4.3.4. Under the assumptions of Theorem 4.2.1, for $\forall x' \in \partial U$ there exists $\delta = \delta(x') > 0$, such that for ε_0 small enough and for all functions u with $\operatorname{supp} u \subset V_{x',\delta}$, the a priori estimate (4.2.20) holds.

Proof. Let l be such that $x' \in U_l$. Denote by $L_{(l)}(y,\varepsilon,\eta)$, $L_{(l)j}(y',\varepsilon,\eta)$, $1 \le j \le r_2+r_3$, the symbols L and L_j in the new coordinates $y = (y',y_n) \in \chi_l(U_l \cap \overline{U})$, respectively. Introduce

$$W_\delta \overset{\text{def}}{=} \{y \in \mathbb{R}^n_+ \mid |y-\chi_l(x')| < \delta\}.$$

Moreover, let ψ, Ψ be C^∞-functions which are identically one on W_δ, $W_{2\delta}$ and have their supports in $W_{2\delta}$, $W_{3\delta}$, respectively. Finally, define the symbols L_l, L_{jl} by

$$L_l = L_{(l)0}(\chi_l(x'),\varepsilon,\widehat{\eta'},\eta_n)$$

$$L_{jl} = L_{(l)j0}(\chi_l(x'),\varepsilon,\widehat{\eta'},\eta_n)$$

Let u be such that $\operatorname{supp} u \subset W_\delta$. Then one obtains from Theorem 3.2.1 after an integration with respect to $\eta' \in \mathbb{R}^{n-1}$:

$$(4.3.8) \qquad ||u||^+_{(s)} = ||\psi u||^+_{(s)} \le$$

$$\le C(||\pi_+ \mathrm{Op}\, L_l l_0(\psi u)||^+_{(s-\nu)} + \sum_{1\le j\le r_2} [\pi_0\, \mathrm{Op}\, L_{jl} l_0(\psi u)]_{(\tau_j)} +$$

$$+ \sum_{r_2<j\le r_2+r_3} [\pi_0\, \mathrm{Op}\, L_{jl} l_0(\psi u)]_{(\sigma_j)}).$$

The symbols $L_{(l)}$, $L_{(l)j}$ can be decomposed as follows:

$$L_{(1)} = L_1 + R_1 + R_2$$

$$L_{(1)j} = L_{j1} + R_{j1} + R_{j2}$$

where

$$R_1(y,\varepsilon,\eta) = L_{(1)0}(y,\varepsilon,\widehat{\eta'},\eta_n)-L_{(1)0}(\chi_1(x'),\varepsilon,\widehat{\eta'},\eta_n)$$

$$R_2(y,\varepsilon,\eta) = L_{(1)}(y,\varepsilon,\eta)-L_{(1)0}(y,\varepsilon,\widehat{\eta'},\eta_n)$$

$$R_{j1}(y',\varepsilon,\eta) = L_{(1)j0}(y',\varepsilon,\widehat{\eta'},\eta_n)-L_{(1)j0}(\chi_1(x'),\varepsilon,\widehat{\eta'},\eta_n)$$

$$R_{j2}(y',\varepsilon,\eta) = L_{(1)j}(y',\varepsilon,\eta)-L_{(1)j0}(y',\varepsilon,\widehat{\eta'},\eta_n).$$

Therefore, one has:

$$(4.3.9) \qquad ||\pi_+\mathrm{Op}\ L_1 l_0(\psi u)||^+_{(s-\nu)} \le$$

$$\le ||\pi_+\mathrm{Op}\ L_{(1)} l_0(\psi u)||^+_{(s-\nu)} + ||\pi_+\mathrm{Op}\ R_1 l_0(\psi u)||_{H_{(s-\nu)}(W_{3\delta})} +$$

$$+ ||\pi_+\mathrm{Op}\ R_1 l_0(\psi u)||_{H_{(s-\nu)}(\mathbb{R}^n_+\setminus W_{3\delta})} + ||\pi_+\ \mathrm{Op}\ R_2 l_0(\psi u)||^+_{(s-\nu)}$$

According to the construction of ψ,Ψ, one has $\psi(y)(1-\Psi)(y) \equiv 0$. Lemma 4.3.1 now implies that the operator $(1-\Psi)R_1\psi$ has at most order $\nu-e_2$. Hence,

$$(4.3.10) \qquad ||\mathrm{Op}\ R_1(\psi u)||_{H_{(s-\nu)}(\mathbb{R}^n_+\setminus W_{3\delta})} \le C||(1-\Psi)\ \mathrm{Op}\ R_1 l_0(\psi u)||^+_{(s-\nu)}$$

$$\le C_\delta||\psi u||^+_{(s-e_2)}.$$

Since the symbol R_1 can be estimated as follows:

$$|R_1(y,\varepsilon,\eta)| \le C\delta\varepsilon^{-\nu_1}\langle\eta\rangle^{\nu_2}\langle\varepsilon\eta\rangle^{\nu_3} \quad \forall y \in W_{3\delta},\ \forall\eta \in \mathbb{R}^n,\ \forall\varepsilon \in (0,\varepsilon_0],$$

one has:

$$(4.3.11) \qquad ||\mathrm{Op}\ R_1 l_0(\psi u)||_{H_{(s-\nu)}(W_{3\delta})} \le C\delta||\psi u||^+_{(s)}$$

with C independent upon δ,ε,u.

Since $|\widehat{\eta'}-\eta'|$ is bounded uniformly with respect to $\eta' \in \mathbb{R}^{n-1}\setminus\{0\}$, R_2 is a symbol of order at most $\nu-e_2$ and

$$(4.3.12)\quad ||\pi_+ \mathrm{Op}\ R_2 l_0(\psi u)||^+_{(s-\nu)} \le C||u||^+_{(s-e_2)} \qquad \forall u \in H_{(s)}.$$

The inequalities (4.3.9)-(4.3.12) yield:

$$||\pi_+ \mathrm{Op}\ L_1 l_0(\psi u)||^+_{(s-\nu)} \le ||\mathrm{Op}\ L_{(1)} l_0(\psi u)||^+_{(s-\nu)} + \\ + C\delta||u||^+_{(s)} + C_\delta||u||^+_{(s-e_2)}.$$

Similarly, one obtains

$$||\pi_+ \mathrm{Op}\ L_{j1} l_0(\psi u)||^+_{(s-\mu_j)} \le ||\mathrm{Op}\ L_{(1)j} l_0(\psi u)||^+_{(s-\mu_j)} + \\ + C\delta||u||^+_{(s)} + C_\delta||u||^+_{(s-e_2)}$$

with C independent upon δ,ε,u. As a consequence of (4.3.8), one finds that

$$||u||^+_{(s)} \le C(||\mathrm{Op}\ L_{(1)} l_0(\psi u)||^+_{(s-\nu)} + \sum_{1\le j\le r_2} [\pi_0 \mathrm{Op}\ L_{(1)j} l_0(\ u)]_{(\tau_j)} + \\ + \sum_{r_2<j\le r_2+r_3} [\pi_0 \mathrm{Op}\ L_{(1)j} l_0(\psi u)]_{(\sigma_j)}) + \\ + C_1\delta||u||^+_{(s)} + C_\delta||u||^+_{(s-e_2)}$$

with C_1 independent upon δ. We now choose $\delta = \frac{1}{2}C_1$. This ends the proof of Proposition 4.3.4. □

<u>Proof of Theorem 4.2.1</u>.

The first inequality in (4.2.20) follows from the trace theorem ([Fr 1]).In order to prove the second one, let $V_{x',\delta(x)}$ be as in Proposition 4.3.4.

Since ∂U is compact, there exist finitely many points $x'_l \in \partial U$, $1 \le l \le r$, such that $\partial U \supset \bigcup_{1\le l\le r} \overline{V}_{x'_l,\delta(x'_l)}$. With $V_l \overset{\mathrm{def}}{=} V_{x'_l,\delta(x'_l)}$, let $V_0 \subset\subset U$ be an open set such that $(V_l)_{0\le l\le r}$ covers U. Moreover, let $(\psi_l)_{0\le l\le r}$ be a partition of unity subordinate to the covering (V_l). One has:

$$||u||_{(s)} \le \sum_{0\le l\le r} ||\psi_l u||_{(s)}.$$

Using the Propositions 4.3.3 and 4.3.4 in order to estimate the terms

$||\psi_l u||_{(s)}$ for $l = 0$ and $l \geq 1$, respectively, one obtains:

$$(4.3.12') \quad ||u||_{(s)} \leq C\Big(\sum_{0\leq l\leq r} ||\pi \text{ Op } L\, l_0(\psi_l u)||_{(s-\nu)} +$$

$$+ \sum_{\substack{1\leq l\leq r \\ 1\leq j\leq r_2}} [\pi_0 \text{ Op } L_j l_0(\psi_l u)]_{(\tau_j)} + \sum_{\substack{1\leq l\leq r \\ r_2<j\leq r_2+r_3}} [\pi_0 \text{ Op } L_j l_0(\psi_l u)]_{(\sigma_j)} +$$

$$+ ||u||_{(s-e_2)}\Big)$$

with C independent upon u,ε.

As a consequence of Lemma 4.3.2, the operator $R = [\text{Op } L, \text{Op } \psi_l]$ has at most order $\nu-e_2$. Hence,

$$(4.3.13) \quad ||\pi \text{ Op} L\, l_0(\psi_l u)||_{(s-\nu)} \leq ||\pi\, \psi_l \text{Op } L\, l_0 u\, ||_{(s-\nu)} + ||\pi\, R\, l_0 u||_{(s-\nu)}$$

$$\leq C(||f||_{(s-\nu)} + ||u||_{(s-e_2)}), \ \forall l,\ 0 \leq l \leq r$$

Similarly, one finds that for $1 \leq l \leq r$,

$$(4.3.14)\begin{cases} [\pi_0 \text{Op } L_j l_0(\psi_l (u)]_{(\tau_j)} \leq C([\phi_j]_{(\tau_j)} + ||u||_{(s-e_2)}), & 1 \leq j \leq r_2, \\ [\pi_0 \text{Op } L_j l_0(\psi_l u)]_{(\sigma_j)} \leq C([\phi_j]_{(\sigma_j)} + ||u||_{(s-e_2)}), & r_2 < j \leq r_2+r_3 \end{cases}$$

The second inequality in (4.2.20) follows from (4.3.12') - (4.3.14). □

The proof of Theorem 4.2.2 is very similar to the one of Theorem 4.2.1.

In order to prove Theorem 4.2.3 stated above, we need several lemmas. In the proofs of these auxiliary results, we are going to use a technique from [K-N].

Lemma 4.3.5. ([Fr-W]). Let $p^{(j)}(x,\varepsilon,\xi) \in L_{(\nu^{(j)})}$, $j = 1,2$, and assume that the reduced symbol $p^{(1)0}$ is identically one. Then one has:

$$(4.3.15) \quad ||\text{Op } p^{(1)} \circ \text{Op } p^{(2)} - \text{Op}(p^{(1)}p^{(2)})||_{\text{Hom}(H_{(s)},H_{(s-\nu^{(1)}-\nu^{(2)})})} \leq C_s\varepsilon$$

where C_s is a constant independent upon ε.

Proof. One has $p^{(j)}(x,\varepsilon,\xi) = p^{(j)'}(x,\varepsilon,\xi) + p_\infty^{(j)}(\varepsilon,\xi)$ where $p^{(j)'}$ is rapidly decreasing for $|x| \to \infty$. Since $p_\infty^{(2)}$ does not depend upon x, the following holds:

$$\text{(4.3.16)} \quad \mathrm{Op}\, p^{(1)} \circ \mathrm{Op}\, p_\infty^{(2)} - \mathrm{Op}(p^{(1)} p_\infty^{(2)}) = 0.$$

Thus, it suffices to show that

$$\text{(4.3.17)} \quad \mathrm{Op}\, p^{(1)'} \circ \mathrm{Op}\, p^{(2)'} - \mathrm{Op}(p^{(1)'} p^{(2)'}) = \varepsilon Q_1$$

$$\text{(4.3.18)} \quad \mathrm{Op}\, p_\infty^{(1)} \circ \mathrm{Op}\, p^{(2)'} - \mathrm{Op}(p_\infty^{(1)} p^{(2)'}) = \varepsilon Q_2,$$

where Q_1 and Q_2 both have at most order $\nu^{(1)}+\nu^{(2)}$.

First, we prove (4.3.17). One has:

$$((\mathrm{Op}\, p^{(1)'} \circ \mathrm{Op}\, p^{(2)'} - \mathrm{Op}(p^{(1)'} p^{(2)'}))u)^\wedge(\xi) =$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} K(\xi,\varepsilon,\eta)\hat{u}(\eta)\,d\eta$$

where

$$\text{(4.3.19)} \quad K(\xi,\varepsilon,\eta) = \int_{\mathbb{R}^n} (\widehat{p^{(1)'}}(\xi-\tau,\varepsilon,\tau) - \widehat{p^{(1)'}}(\xi-\tau,\varepsilon,\eta))\widehat{p^{(2)'}}(\tau-\eta,\varepsilon,\eta)\,d\tau$$

and the hat denotes the Fourier transform with respect to the first variable. By Schur's lemma on the boundedness of integral operators in L^2, in order to prove (4.3.17) it is sufficient to show that with $\nu = \nu^{(1)}+\nu^{(2)}$:

$$\text{(4.3.20)} \quad \varepsilon^{\nu_1} \int \langle\xi\rangle^{s_2-\nu_2} \langle\varepsilon\xi\rangle^{s_3-\nu_3} \langle\eta\rangle^{-s_2} \langle\varepsilon\eta\rangle^{-s_3} |K(\xi,\varepsilon,\eta)|\,d\xi \le C\varepsilon$$

$$\text{(4.3.21)} \quad \varepsilon^{\nu_1} \int \langle\xi\rangle^{s_2-\nu_2} \langle\varepsilon\xi\rangle^{s_3-\nu_3} \langle\eta\rangle^{-s_2} \langle\varepsilon\eta\rangle^{-s_3} |K(\xi,\varepsilon,\eta)|\,d\eta \le C\varepsilon$$

with some constant C which does not depend upon ε, η and ξ. With the symbol b defined by the formula

$$b(x,\varepsilon,\xi) = \varepsilon^{\nu_1^{(1)}} \langle\varepsilon\xi\rangle^{-\nu_3^{(1)}} p^{(1)}(x,\varepsilon,\xi) - 1 \in L_{(-1,1,-1)},$$

one has:

$$|D_x^\alpha(p^{(1)'}(x,\varepsilon,\tau)-p^{(1)'}(x,\varepsilon,\eta))| \le |D_x^\alpha(b'(x,\varepsilon,\tau)-b'(x,\varepsilon,\eta))\varepsilon^{-\nu_1^{(1)}}<\varepsilon\tau>^{\nu_3^{(1)}}| + $$
$$+ |D_x^\alpha b'(x,\varepsilon,\eta)\varepsilon^{-\nu_1^{(1)}}(<\varepsilon\tau>^{\nu_3^{(1)}}-<\varepsilon\eta>^{\nu_3^{(1)}})|.$$

Therefore, using also (4.1.2), one obtains:

$$(4.3.22)\quad |\widehat{p^{(1)'}}(\zeta,\varepsilon,\tau)-\widehat{p^{(1)'}}(\zeta,\varepsilon,\eta)| \le C_k<\tau-\eta>^m\varepsilon^{1-\nu_1^{(1)}}<\varepsilon\tau>^{\nu_3^{(1)}}<\zeta>^{-k}$$

$$\forall\zeta,\varepsilon,\eta,\ \forall k \ge 0$$

with $m = 2+|\nu_3^{(1)}|$. Similarly, one has:

$$(4.3.23)\quad |p_\infty^{(1)}(\varepsilon,\tau)-p_\infty^{(1)}(\varepsilon,\eta)| \le C<\tau-\eta>^m\varepsilon^{1-\nu_1^{(1)}}<\varepsilon\tau>^{\nu_3^{(1)}}$$

Using (4.3.22) and the estimate

$$|\widehat{p^{(2)'}}(\zeta,\varepsilon,\eta)| \le C_k\ \varepsilon^{-\nu_1^{(2)}}<\eta>^{\nu_2^{(2)}}<\varepsilon\eta>^{\nu_3^{(2)}}<\zeta>^{-k},$$

which follows from the definition of $L_{(\nu)}$, one can estimate the integral on the left hand side of (4.3.20) in the following fashion:

$$(4.3.24)\quad \varepsilon^{\nu_1}\int<\xi>^{s_2-\nu_2}<\varepsilon\xi>^{s_3-\nu_3}<\eta>^{-s_2}<\varepsilon\eta>^{-s_3}|K(\xi,\varepsilon,\eta)|d\xi \le$$
$$\le C_k\int<\xi>^{-\nu_2}<\varepsilon\xi>^{-\nu_3}<\xi-\eta>^{|s_2|+|s_3|}\int<\tau-\eta>^m\varepsilon<\varepsilon\tau>^{\nu_3^{(1)}}<\xi-\tau>^{-k}$$
$$<\eta>^{\nu_2^{(2)}}<\varepsilon\eta>^{\nu_3^{(2)}}<\tau-\eta>^{-k}d\tau\ d\xi$$
$$\le C_k\varepsilon\int\int<\tau-\xi>^{m_2-k}d\xi<\eta-\tau>^{m_1-k}d\tau,$$

where

$$m_1 = m+|s_2|+|s_3|+|\nu_2^{(2)}|+|\nu_3^{(2)}|$$
$$m_2 = |\nu_3^{(1)}|+|s_2|+|s_3|+|\nu_2^{(2)}|+|\nu_3^{(2)}|.$$

We choose k to be sufficiently large in order to guarantee the convergence of the integral in the right hand side of (4.3.24) and that ends the proof of (4.3.20). The same argument can be used in order to prove (4.3.21). Hence, we have proved (4.3.17).

Using (4.3.23) instead of (4.3.22), one gets (4.3.20), (4.3.21) with K replaced by the function

$$K_\infty(\xi,\varepsilon,\eta) = (p_\infty^{(1)}(\varepsilon,\xi)-p_\infty^{(1)}(\varepsilon,\eta))\widehat{p^{(2)\prime}}(\xi-\eta,\varepsilon,\eta).$$

This yields (4.3.18). □

In order to prove Theorem 4.2.3, Lemma 4.3.5 will be generalized to Wiener-Hopf operators with variable coefficients in a half space. Let $R^{(1)}$, $R^{(2)}$ be two Wiener-Hopf symbols of the form (4.1.7) such that the reduced operator of $\mathrm{Op}\, R^{(1)}$ is the identity operator and such that $s^{(2)} = r^{(1)}$. Then in the following lemmas, it will be shown that the operator

$$\mathrm{Op}\, R^{(1)} \circ \mathrm{Op}\, R^{(2)} - \mathrm{Op}\, (R^{(1)} \circ R^{(2)})$$

is of order $O(\varepsilon^{\gamma_0}|\ln \varepsilon|)$ in appropriate spaces, where γ_0 is the order of approximation.

Lemma 4.3.6. If the reduced operator of the Poisson operator $\pi_+ \,\mathrm{Op}\, k \in \mathrm{Hom}\,(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(s)}(\mathbb{R}^n_+))$ is zero, and if $\mathrm{Op}\, q \in \mathrm{Hom}(H_{(\tau)}(\mathbb{R}^{n-1}), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is a pseudodifferential operator on the boundary, then

$$||\pi_+ \,\mathrm{Op}\, k \circ \mathrm{Op}\, q - \mathrm{Op}\, (k\, q)||_{\mathrm{Hom}(H_{(\tau)}(\mathbb{R}^{n-1}),H_{(s)}(\mathbb{R}^n_+))} \le C\, \varepsilon^{\gamma_0}|\ln \varepsilon|$$

Proof. One has:

$$((\pi_+ \,\mathrm{Op}\, k \circ \mathrm{Op}\, q - \mathrm{Op}(k\circ q)\phi)^\wedge(\xi) = (2\pi)^{\frac{1}{2}-n} \int_{\mathbb{R}^{n-1}} K(\xi,\varepsilon,\eta')\hat{\phi}(\eta')d\eta'$$

where K is given by

$$K(\xi,\varepsilon,\eta') = \int_{\mathbb{R}^{n-1}} (\hat{k}(\xi'-\zeta',\varepsilon,\zeta',\xi_n)-\hat{k}(\xi'-\zeta',\varepsilon,\eta',\xi_n))\hat{q}(\zeta'-\eta',\varepsilon,\eta')d\zeta'.$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.6. □

Lemma 4.3.7. If the reduced operator of the pseudodifferential operator $\pi_+ \,\mathrm{Op}\, p\, l_0 \in \mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(t)}(\mathbb{R}^n_+))$ is the identity operator, and if

$\pi_+ \text{ Op } k \in \text{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(s)}(\mathbb{R}^n_+))$ is a Poisson operator, then

$$\|\pi_+ \text{ Op } p\, l_0 \circ \pi_+ \text{ Op } k - \pi_+ \text{ Op}(p \circ k)\|_{\text{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(t)}(\mathbb{R}^n_+))} \le C\, \varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. Let $p = p' + p_\infty$, $k = k' + k_\infty$ with p', k' rapidly decreasing as $|x| \to \infty$ and $|x'| \to \infty$, respectively. One has:

$$((\pi_+ \text{ Op } p' l_0 \pi_+ \text{Op } k' - \pi_+ \text{ Op}(p' \circ k'))\phi)\hat{}\,(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} K(\xi, \varepsilon, \eta') \hat{\phi}(\eta') d\eta'$$

where

$$K(\xi, \varepsilon, \eta') = \Pi^+_{\xi_n} \int_{\mathbb{R}^n} (\widehat{p'}(\xi-\zeta, \varepsilon, \zeta', \xi_n) - \widehat{p'}(\xi-\zeta, \varepsilon, \eta', \xi_n))\widehat{k'}(\zeta'-\eta', \varepsilon, \eta', \xi_n) d\zeta.$$

Estimating the function $|K|$ similarly as it was done in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.7 with p, k replaced by p', k'. □

Lemma 4.3.8. If the reduced operator of the singular Green operator $\pi_+ \text{ Op } g\, l_0 \in \text{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(t)}(\mathbb{R}^n_+))$ is zero, and if $\pi_+ \text{ Op } k \in \text{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(s)}(\mathbb{R}^n_+))$ is a Poisson operator, then

$$\|\pi_+ \text{ Op } g\, l_0\, \pi_+ \text{ Op } k - \pi_+ \text{ Op}(g \circ k)\|_{\text{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}), H_{(t)}(\mathbb{R}^n_+)} \le C\, \varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. One has:

$$((\pi_+ \text{ Op } g\, l_0\, \pi_+ \text{ Op } k - \pi_+ \text{ Op } g \circ k)\psi)\hat{}\,(\xi) = (2\pi)^{-n} \int K(\xi, \varepsilon, \eta') \tilde{\psi}(\eta') d\eta'$$

where

$$K(\xi, \varepsilon, \eta') = \int_{\mathbb{R}^{n-1}} \int_{\gamma} (\hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n) - \hat{g}(\xi'-\zeta', \varepsilon, \eta', \xi_n, \zeta_n))\hat{k}(\zeta'-\eta', \varepsilon, \eta', \zeta_n) d\zeta$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.8. □

Lemma 4.3.9. If the reduced operator of the trace operator $\pi_0 \text{ Op } t\, 1_0 \in \text{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is zero, and if $\pi_+ \text{ Op } p\, 1_0 \in \text{Hom}(H_{(t)}(\mathbb{R}^n_+), H_{(s)}(\mathbb{R}^n_+))$ is a pseudodifferential operator, then

$$||\pi_0 \text{ Op } t\, 1_0\, \pi_+ \text{ Op } p\, 1_0 - \pi_0 \text{ Op } (t\circ p) 1_0||_{\text{Hom}(H_{(t)}(\mathbb{R}^n_+),\, H_{(\sigma)}(\mathbb{R}^{n-1}))} \leq C\varepsilon^{\gamma_0}|\ln \varepsilon|$$

Proof. One has:

$$((\pi_0 \text{ Op } t\, 1_0 \pi_+ \text{ Op } p - \pi_0 \text{ Op}(t\circ p))(1_0 u))\hat{}\,(\xi') = (2\pi)^{\frac{1}{2}-n} \int K(\xi',\varepsilon,\eta)\widehat{1_0 u}(\eta)d\eta$$

where

$$K(\xi',\varepsilon,\eta) = \Pi^-_{\eta_n} \int_{\mathbb{R}^n} (\hat{t}(\xi'-\zeta',\varepsilon,\zeta) - \hat{t}(\xi'-\zeta',\varepsilon,\eta))\hat{p}(\zeta-\eta,\varepsilon,\eta)d\zeta$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.8. □

Lemma 4.3.10. If the reduced operator of the trace operator $\pi_0 \text{ Op } t\, 1_0 \in \text{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is zero, and if $\pi_+ \text{ Op } g\, 1_0 \in \text{Hom}(H_{(t)}(\mathbb{R}^n_+), H_{(s)}(\mathbb{R}^n_+))$ is a singular Green operator, then

$$||\pi_0 \text{ Op } t\, 1_0\, \pi_+ \text{ Op } g\, 1_0 - \pi_0 \text{ Op}(t\circ g) 1_0||_{\text{Hom}(H_{(t)}(R^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))} \leq C\, \varepsilon^{\gamma_0}|\ln \varepsilon|$$

Proof. One has:

$$(((\pi_0 \text{ Op } t\, 1_0\, \pi_+ \text{ Opg} - \pi_0 \text{ Op}(t\circ g))(1_0 u))\hat{}\,(\xi') = (2\pi)^{\frac{1}{2}-n} \int K(\xi',\varepsilon,\eta)\widehat{1_0 u}(\eta)d\eta$$

where

$$K(\xi',\varepsilon,\eta) = \int_{\mathbb{R}^{n-1}} \int_{\gamma} (\hat{t}(\xi'-\zeta',\varepsilon,\zeta)-\hat{t}(\xi'-\zeta',\varepsilon,\eta',\zeta_n))\hat{g}(\zeta'-\eta',\varepsilon,\eta',\zeta_n,\eta_n)d\zeta' d\zeta_n$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.10. □

Lemma 4.3.11. If the pseudodifferential operator $\mathrm{Op}\, q \in \mathrm{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}),$ $H_{(\tau)}(\mathbb{R}^{n-1}))$ satisfies $\mathrm{Op}\, q^0 = 0$ or $\mathrm{Op}\, q^0 = \mathrm{Id}$, and if $\pi_0\, \mathrm{Op}\, t \in \mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is a trace operator, then

$$\|\mathrm{Op}\, q\, \pi_0\, \mathrm{Op}\, t - \mathrm{Op}(q \circ t)\|_{\mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\tau)}(\mathbb{R}^{n-1}))} \le C\, \varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. One has:

$$((\mathrm{Op}\, q\, \pi_0\, \mathrm{Op}\, t - \mathrm{Op}(q \circ t))(1_0 u))\hat{}\,(\xi') = (2\pi)^{1-n} \int K(\xi',\varepsilon,\eta) \widehat{1_0 u}(\eta) d\eta$$

where

$$K(\xi',\varepsilon,\eta) = \int_{\mathbb{R}^{n-1}} (\hat{q}(\xi'-\zeta',\varepsilon,\zeta') - \hat{q}(\xi'-\zeta',\varepsilon,\eta'))\hat{t}(\zeta'-\eta',\varepsilon,\eta) d\zeta'$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.11. □

Lemma 4.3.12. If the reduced operator of the trace operator $\pi_0\, \mathrm{Op}\, t\, 1_0 \in$ $\mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+), H_{(\sigma)}(\mathbb{R}^{n-1}))$ is zero, and if $\pi_+\, \mathrm{Op}\, k \in \mathrm{Hom}(H_{(\tau)}(\mathbb{R}^{n-1}),$ $H_{(s)}(\mathbb{R}^n_+))$ is a Poisson operator, then

$$\|\pi_0\, \mathrm{Op}\, t\, 1_0\, \pi_+\, \mathrm{Op}\, k - \mathrm{Op}\, t \circ k\|_{\mathrm{Hom}(H_{(\tau)}(\mathbb{R}^{n-1}), H_{(\sigma)}(\mathbb{R}^{n-1}))} \le C\, \varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. One has:

$$((\pi_0\, \mathrm{Op}\, t\, 1_0\, \pi_+\, \mathrm{Op}\, k - \mathrm{Op}\, t \circ k)\psi)\hat{}\,(\xi') = (2\pi)^{\frac{1}{2}-n} \int K(\xi',\varepsilon,\eta') \hat{\psi}(\eta') d\eta'$$

where

$$K(\xi',\varepsilon,\eta') = \int_{\mathbb{R}^{n-1}} \int_{\gamma} (\hat{t}(\xi'-\zeta',\varepsilon,\zeta) - \hat{t}(\xi'-\zeta',\varepsilon,\eta',\zeta_n))\hat{k}(\zeta'-\eta',\eta',\zeta_n) d\zeta' d\zeta_n$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.12. □

Lemma 4.3.13. If the reduced operator of the Poisson operator

$\pi_+ \; \mathrm{Op}\; k \in \mathrm{Hom}(H_{(\sigma)}(\mathbb{R}^{n-1}),\; H_{(s)}(\mathbb{R}^n_+))$ is zero, and if

$\pi_0 \; \mathrm{Op}\; t \; 1_0 \in \mathrm{Hom}(H_{(s)}(\mathbb{R}^n_+),\; H_{(\sigma)}(\mathbb{R}^{n-1}))$ is a trace operator, then

$$||\pi_+ \; \mathrm{Op}\; k \; \pi_0 \; \mathrm{Op}\; t \; 1_0 - \pi_+ \; \mathrm{Op}(k \circ t) 1_0||_{\mathrm{Hom}(H_{(t)}(\mathbb{R}^n_+),H_{(s)}(\mathbb{R}^n_+))} \le C \, \varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. One has:

$$((\pi_+ \; \mathrm{Op}\; k \; \pi_0 \; \mathrm{Op}\; t - \pi_+ \; \mathrm{Op}\; k \circ t)(1_0 u))\hat{}\,(\xi) = (2\pi)^{\frac{1}{2}-n} \int K(\xi,\varepsilon,\eta) \widehat{1_0 u}(\eta) d\eta$$

where

$$K(\xi,\varepsilon,\eta) = \int_{\mathbb{R}^{n-1}} (\hat{k}(\xi'-\zeta',\varepsilon,\zeta',\xi_n) - \hat{k}(\xi'-\zeta',\varepsilon,\eta',\xi_n))\hat{t}(\zeta'-\eta',\eta)d\zeta'$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.13. □

Lemma 4.3.14. If the pseudodifferential operators

$\pi_+ \; \mathrm{Op}\; p \; 1_0 \in \mathrm{Hom}(H_{(s_2)}(\mathbb{R}^n_+),\; H_{(s_3)}(\mathbb{R}^n_+))$ satisfies $\mathrm{Op}\; p^0 = 0$ or $\mathrm{Op}\; p^0 = \mathrm{Id}$, and if $\pi_+ \; \mathrm{Op}\; g \; 1_0 \in \mathrm{Hom}(H_{(s_3)}(\mathbb{R}^n_+),\; H_{(s_2)}(\mathbb{R}^n_+))$ is a singular Green operator, then

$$||\pi_+ \; \mathrm{Op}\; p \; 1_0 \; \pi_+ \; \mathrm{Op}\; g \; 1_0 - \pi_+ \; \mathrm{Op}(p \circ g) 1_0||_{(\mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+),H_{(s_3)}(\mathbb{R}^n_+))} \le C\varepsilon^{\gamma_0} |\ln \varepsilon|$$

Proof. One has:

$$((\pi_+ \; \mathrm{Op}\; p \; 1_0 \; \pi_+ \; \mathrm{Op}\; g - \pi_+ \; \mathrm{Op}(p \circ g))(1_0 u))\hat{}\,(\xi) = (2\pi)^{-n} \int K(\xi,\varepsilon,\eta) \widehat{1_0 u}(\eta) d\eta$$

where

$$K(\xi,\varepsilon,\eta) = \Pi^+_{\xi_n} \int_{\mathbb{R}^{n-1}} \int_{\gamma} (\hat{p}(\xi-\zeta,\varepsilon,\zeta) - \hat{p}(\xi-\zeta,\varepsilon,\eta',\zeta_n))\tilde{g}(\zeta'-\eta',\varepsilon,\eta',\zeta_n,\eta_n)d\zeta$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.14. □

Lemma 4.3.15. If the reduced operator of the singular Green operator $\pi_+ \mathrm{Op}\, g\, 1_0 \in \mathrm{Hom}(H_{(s_2)}(\mathbb{R}^n_+), H_{(s_3)}(\mathbb{R}^n_+))$ is zero and if $\pi_+ \mathrm{Op}\, p\, 1_0 \in \mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+), H_{(s_2)}(\mathbb{R}^n_+))$ is a pseudodifferential operator, then

$$\| \pi_+ \mathrm{Op}\, g\, 1_0 \pi_+ \mathrm{Op}\, p\, 1_0 - \pi_+ \mathrm{Op}(g \circ p) 1_0 \|_{\mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+), H_{(s_3)}(\mathbb{R}^n_+))} \leq C \varepsilon^{\gamma_0} |\ln \varepsilon| .$$

Proof. Since $\int_\gamma f_1 (\Pi^+ f_2) = \int_\gamma (\Pi^- f_1) f_2$ (see [B.d.M]), one has:

$$(\pi_+ \mathrm{Op}\, g\, 1_0 \pi_+ \mathrm{Op}\, p\, 1_0 u)^\wedge (\xi) =$$

$$= (2\pi)^{-n} \iint \hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n) \Pi^+_{\zeta_n} \hat{p}(\zeta-\eta, \varepsilon, \eta)\, \widehat{1_0 u}(\eta) d\zeta\, d\eta$$

$$= (2\pi)^{-n} \iint (\Pi^-_{\zeta_n} \hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n)) \hat{p}(\zeta-\eta, \varepsilon, \eta)\, \widehat{1_0 u}(\eta) d\zeta\, d\eta$$

$$= (2\pi)^{-n} \iint \hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n) \hat{p}(\zeta-\eta, \varepsilon, \eta)\, \widehat{1_0 u}(\eta) d\zeta\, d\eta$$

$$= (2\pi)^{-n} \iint \hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n) \hat{p}(\zeta-\eta, \varepsilon, \eta) \Pi^+_{\eta_n} \widehat{1_0 u}(\eta) d\zeta\, d\eta$$

$$= (2\pi)^{-n} \iint \Pi^-_{\eta_n} (\hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \zeta_n) \hat{p}(\zeta-\eta, \varepsilon, \eta))\, \widehat{1_0 u}(\eta) d\zeta\, d\eta .$$

Thus, one obtains

$$((\pi_+ \mathrm{Op}\, g\, 1_0 \pi_+ \mathrm{Op}\, p - \pi_+ \mathrm{Op}(g \circ p))(1_0 u))^\wedge (\xi) = (2\pi)^{-n} \int K(\xi, \varepsilon, \eta) \widehat{1_0 u}(\eta) d\eta$$

where

$$K(\xi, \varepsilon, \eta) = \Pi^-_{\eta_n} \int (\hat{g}(\xi'-\zeta', \varepsilon, \zeta', \xi_n, \eta_n) - \hat{g}(\xi'-\zeta', \varepsilon, \eta', \xi_n, \eta_n)) \hat{p}(\zeta-\eta, \varepsilon, \eta) d\zeta .$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.15. □

Lemma 4.3.16. If the reduced operator of the singular Green operator $\pi_+ \mathrm{Op}\, g_1 1_0 \in \mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+), H_{(s_2)}(\mathbb{R}^n_+))$ is zero and if $\pi_+ \mathrm{Op}\, g_2 1_0$ $\in \mathrm{Hom}(H_{(s_2)}(\mathbb{R}^n_+), H_{(s_3)}(\mathbb{R}^n_+))$ is another singular Green operator, then

$$||\pi_+ \mathrm{Op}\, g_1 1_0 \pi_+ \,\mathrm{Op}\, g_2 1_0 - \pi_+ \mathrm{Op}(g_1 \circ g_2) 1_0||_{\mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+), H_{(s_3)}(\mathbb{R}^n_+))} \le C^{\gamma_0} |\ln \varepsilon|.$$

<u>Proof</u>. One has:

$$((\pi_+ \mathrm{Op}\, g_1 1_0 \pi_+ \mathrm{Op}\, g_2 - \pi_+ \mathrm{Op}(g_1 \circ g_2))(1_0 u))\hat{\ }(\xi) = (2\pi)^{-n} \int K(\xi,\varepsilon,\eta)\, \widehat{1_0 u}(\eta) d\eta$$

where

$$K(\xi,\varepsilon,\eta) = \int_{\mathbb{R}^{n-1}} \int_{\gamma} (\hat{g}_1(\xi'-\zeta',\varepsilon,\zeta',\xi_n,\zeta_n) - \hat{g}_1(\xi'-\zeta',\varepsilon,\eta',\xi_n,\zeta_n)) \hat{g}_2(\zeta'-\eta',\varepsilon,\eta',\zeta_n,\eta_n) d\zeta$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.16. □

<u>Lemma 4.3.17</u>. For pseudodifferential symbols p, one has:

$$\Pi^{\pm}_{\xi_n} \int \hat{p}(\xi-\eta,\varepsilon,\eta)\, \widehat{1_0 u}(\eta) d\eta = \int \Pi^{\pm}_w (\hat{p}(\xi-\eta,\varepsilon,\eta',w)\, \widehat{1_0 u}(\eta',w))|_{w=\eta_n} d\eta.$$

<u>Proof</u>. One has:

$$\Pi^{+}_{\xi_n} \int \hat{p}(\xi-\eta,\varepsilon,\eta)\, \widehat{1_0 u}(\eta) d\eta$$
$$= (2\pi i)^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}} (\xi_n-\rho-i0)^{-1}\, \hat{p}(\xi'-\eta',\rho-\eta_n,\varepsilon,\eta) d\rho\, \widehat{1_0 u}(\eta) d\eta.$$

Using the integration variables $\tilde{\eta}_n = \xi_n+\eta_n-\rho$, $\tau = \eta_n$ instead of ρ,η_n, one obtains

$$\Pi^{+}_{\xi_n} \int \hat{p}(\xi-\eta,\varepsilon,\eta)\, \widehat{1_0 u}(\eta) d\eta =$$
$$= (2\pi i)^{-1} \int\int (\tilde{\eta}_n-\tau-i0)^{-1}\, \hat{p}(\xi'-\eta',\xi_n-\tilde{\eta}_n,\varepsilon,\eta',\tau)\, \widehat{1_0 u}(\eta',\tau) d\eta' d\tilde{\eta}_n\, d\tau$$
$$= (2\pi i)^{-1} \int (\Pi^{+}_w\, \hat{p}(\xi-\eta,\varepsilon,\eta',w)\, \widehat{1_0 u}(\eta',w))|_{w=\eta_n}\, d\eta.$$

For Π^+ replaced by Π^-, the claim of Lemma 4.3.17 is proved similarly. □

Following [B.d.M], we introduce the singular Green symbol $L(p_1,p_2)$, where p_1,p_2 are pseudodifferential symbols, by

$$L(p_1,p_2)(\varepsilon,\xi',\xi_n,\eta_n) =$$

$$= \Pi^+_{\xi_n}\Pi^-_{\eta_n}(p_1^+(\varepsilon,\xi',\xi_n)-p_1^+(\varepsilon,\xi',\eta_n))(p_2^-(\varepsilon,\xi',\xi_n)-p_2^-(\varepsilon,\xi',\eta_n))(i\xi_n-i\eta_n)^{-1}$$

with $p_1^+ = \Pi^+_{\xi_n}p_1$, $p_2^- = \Pi^-_{\xi_n}p_2$. One has (see [B.d.M.]):

$$(\pi_+\mathrm{Op}\ L\ 1_0u)\hat{}(\xi) = \Pi^+_{\xi_n}(p_1p_2)(\varepsilon,\xi)\widehat{1_0u}(\xi) - \Pi^+_{\xi_n}p_1(\varepsilon,\xi)\Pi^+_{\xi_n}p_2(\varepsilon,\xi)\widehat{1_0u}(\xi)$$
$$= \Pi^+_{\xi_n}(p_1(\varepsilon,\xi)\Pi^-_{\xi_n}p_2(\varepsilon,\xi)\widehat{1_0u}(\xi)).$$

<u>Lemma 4.3.18</u>. If the reduced operator of the pseudodifferential operator $\pi_+\mathrm{Op}\ p_1 1_0 \in \mathrm{Hom}(H_{(s_2)}(\mathbb{R}^n_+),H_{(s_3)}(\mathbb{R}^n_+))$ is identity and if $\pi_+\mathrm{Op}\ p_2 1_0 \in \mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+),H_{(s_2)}(\mathbb{R}^n_+))$ is another pseudodifferential operator, then

$$\pi_+\mathrm{Op}\,a\,1_0 \overset{\mathrm{def}}{=} \pi_+\mathrm{Op}\ p_1 1_0\pi_+\mathrm{Op}\ p_2 1_0 - \pi_+\mathrm{Op}(p_1\ p_2)1_0 + \pi_+\mathrm{Op}\ L(p_1,p_2)1_0$$

has the following property:

$$||\pi_+\mathrm{Op}\,a\,1_0||_{\mathrm{Hom}(H_{(s_1)}(\mathbb{R}^n_+),H_{(s_3)}(\mathbb{R}^n_+))} \le C\varepsilon^{\gamma_0}|\ln\varepsilon|.$$

<u>Proof</u>. One has:

$$(2\pi)^n(\pi_+\mathrm{Op}\ p_1 1_0\pi_+\mathrm{Op}\,p_2 - \pi_+\mathrm{Op}(p_1\ p_2))(1_0u)\hat{}(\xi) =$$

$$= \Pi^+_{\xi_n}\int\hat{p}_1(\xi-\eta,\varepsilon,\eta)\Pi^+_{\eta_n}\int\hat{p}_2(\eta-\zeta,\varepsilon,\zeta)\ \widehat{1_0u}(\zeta)d\zeta\ d\eta\ -$$
$$-\ \Pi^+_{\xi_n}\int\hat{p}_1(\xi-\eta,\varepsilon,\zeta)\hat{p}_2(\eta-\zeta,\varepsilon,\zeta)\ \widehat{1_0u}(\zeta)d\zeta\ d\eta$$

$$= (\pi_+\mathrm{Op}(a_1-a_2)1_0u)\hat{}(\xi),$$

where

$$(\pi_+\mathrm{Op}\ a_1 1_0u)\hat{}(\xi) = \int K(\xi,\varepsilon,\zeta)\ \widehat{1_0u}(\zeta)d\zeta$$

$$(\pi_+\mathrm{Op}\ a_2 1_0u)\hat{}(\xi) = \Pi^+_{\xi_n}\iint\hat{p}_1(\xi-\eta,\varepsilon,\zeta)\Pi^-_{\eta_n}\hat{p}_2(\eta-\zeta,\varepsilon,\zeta)\ \widehat{1_0u}(\zeta)d\zeta\ d\eta$$

with

$$K(\xi,\varepsilon,\zeta) = \Pi^+_{\xi_n}\int(\hat{p}_1(\xi-\eta,\varepsilon,\eta)-\hat{p}_1(\xi-\eta,\varepsilon,\zeta))\Pi^+_{\eta_n}\hat{p}_2(\eta-\zeta,\varepsilon,\zeta)d\eta.$$

Lemma 4.3.17 yields:

$$(\pi_+ \mathrm{Op}\ a_2 l_0 u)\hat{}\,(\xi) = \iint (\Pi_w^+ \hat{p}_1(\xi-\eta,\varepsilon,\zeta',w)\Pi_w^- \hat{p}_2(\eta-\zeta,\varepsilon,\zeta',w) l_0 u(\zeta',w))\big|_{w=\xi_n} d\zeta\, d\eta$$

$$= (2\pi)^n (\pi_+ \mathrm{Op}\ L(p_1,p_2) l_0 u)\hat{}\,(\xi).$$

Estimating the function $|K|$ similarly as in the proof of Lemma 4.3.5, one proves the claim of Lemma 4.3.18. □

Let R^ε, B^ε be Wiener-Hopf symbols such that $\mathrm{Op}\ B^\varepsilon \in \mathrm{Hom}(H^{(1)},H^{(2)})$ and $\mathrm{Op}\ R^\varepsilon \in \mathrm{Hom}(H^{(2)},H^{(3)})$.

<u>Proposition 4.3.19</u>. If $\mathrm{Op}\ R^0$ is the identity operator, and if γ_0 is the order of approximation, then

$$\|\mathrm{Op}\ R^\varepsilon \circ \mathrm{Op}\ B^\varepsilon - \mathrm{Op}(R^\varepsilon \circ B^\varepsilon)\|_{\mathrm{Hom}(H^{(1)},H^{(3)})} \le C\varepsilon^{\gamma_0}|\ln\,\varepsilon|$$

with a constant C independent upon ε.

<u>Proof</u>. This statement follows from the Lemmas 4.3.5 - 4.3.18. □

Let now C^ε, R^ε be Wiener-Hopf symbols such that $\mathrm{Op}\ R^\varepsilon \in \mathrm{Hom}(H^{(1)},H^{(2)})$ and $\mathrm{Op}\ C^\varepsilon \in \mathrm{Hom}(H^{(2)},H^{(3)})$. The proof of the following statement is similar to the one of Proposition 4.3.19.

<u>Proposition 4.3.20</u>. If $\mathrm{Op}\ R^0$ is the identity operator, and if γ_0 is the order of approximation, then

$$\|\mathrm{Op}\ C^\varepsilon \circ \mathrm{Op}\ R^\varepsilon - \mathrm{Op}(C^\varepsilon \circ R^\varepsilon)\|_{\mathrm{Hom}(H^{(1)},H^{(3)})} \le C\varepsilon^{\gamma_0}|\ln\,\varepsilon|$$

with a constant C independent upon ε.

<u>Proof of Theorem 4.2.3</u>. Let A^ε, A^0, R_l^ε, R_r^ε, S_l^ε, S_r^ε be the symbols of the Wiener-Hopf operators $\mathfrak{A}^\varepsilon, \mathfrak{A}^0$, $\mathcal{R}_l^\varepsilon$, $\mathcal{R}_r^\varepsilon$, $\mathcal{S}_l^\varepsilon$, $\mathcal{S}_r^\varepsilon$, respectively. First, the inequality (4.2.36) will be proved. Since $\mathrm{Op}\ R_l^0$, $\mathrm{Op}\ R_r^0$ are the identity

operators, the Propositions 4.3.19, 4.3.20 yield:

$$
\begin{aligned}
R_l^\varepsilon \mathfrak{A}^0 R_r^\varepsilon &= \operatorname{Op}(R_l^\varepsilon \circ A^0) \circ \operatorname{Op} R_r^\varepsilon + \varepsilon^{\gamma_0} |\ln \varepsilon| \, Q_1 \\
&= \operatorname{Op}(R_l^\varepsilon \circ A^0 \circ R_r^\varepsilon) + \varepsilon^{\gamma_0} |\ln \varepsilon| \, Q_2 \\
&= \mathfrak{A}^\varepsilon - \operatorname{Op}(A^\varepsilon - R_l^\varepsilon \circ A^0 \circ R_r^\varepsilon) + \varepsilon^{\gamma_0} |\ln \varepsilon| \, Q_2
\end{aligned}
$$

with operators $Q_1, Q_2 \in \operatorname{Hom}(\mathcal{H},\mathcal{K})$. Theorem 3.3.1 now implies

$$
||\operatorname{Op}(A^\varepsilon - R_l^\varepsilon \circ A^0 \circ R_r^\varepsilon)||_{\operatorname{Hom}(\mathcal{H},\mathcal{K})} \le C\varepsilon^{\gamma_0} |\ln \varepsilon|
$$

with C independent upon ε. This proves (4.2.36).

The inequalities (4.2.37), (4.2.38) follow from the Propositions 4.3.19, 4.3.20 and from the fact that S_l^ε, S_r^ε are the inverse symbols of R_l^ε, R_r^ε. Finally, (4.2.39) follows from (4.2.36) - (4.2.38). $\square$

5. EXAMPLES

5.1. Stability results.

The purpose of this section is to illustrate the Theorems 4.2.1 and 4.2.2 by various examples. Let U, l_0, π, π_0 be as in section 4.2.

Example 5.1.1. Consider the following Wiener-Hopf equation

$$(5.1.1) \qquad \pi \frac{1-\varepsilon^2\Delta}{1-2\varepsilon^2\Delta} l_0 u(x) = f(x), \quad x \in \mathcal{U}$$

where $f \in H_{(s)}(\mathcal{U})$ is given and $u \in H_{(s)}(\mathcal{U})$ is sought. Since the condition in Definition 4.1.3 is satisfied with $k_0 = 0$, Theorem 4.2.1 implies that for $-\frac{1}{2} < s_2 < \frac{1}{2}$, $-\frac{1}{2} < s_2+s_3$, the solution u satisfies the a priori estimate

$$||u||_{(s)} \sim ||f||_{(s)} + ||u||_{(s-e_2)}.$$

Moreover, since the reduced problem of (5.1.1) has a unique solution, Theorem 4.2.3 yields the equivalency $||u||_{(s)} \sim ||f||_{(s)}$ uniformly with respect to ε.

Example 5.1.2. Let $p \in \mathcal{L}_{(0,0,0)}$ be an elliptic symbol of order (0,0,0) which in a neighbourhood of ∂U is given in terms of the cotangential and conormal variables ξ', ξ_N by $p(\varepsilon,\xi) = (1+2\varepsilon^2|\xi'|^2+\varepsilon^2\xi_N^2)^{-1}(1+\varepsilon^2|\xi'|^2+\varepsilon^2\xi_N^2)$.

Consider the Wiener-Hopf equation

$$(5.1.2) \qquad \pi_+ \operatorname{Op} p\, l_0 u(x) = f(x), \quad x \in U,$$

where $f \in H_{(s)}(U)$ is given and $u \in H_{(s)}(U)$ is sought. The condition in Definition 4.1.3 is satisfied with $k_0 = 2$. Therefore, Theorem 4.2.1 implies that for $-\frac{1}{2} < s_2 < \frac{5}{2}$, $-\frac{1}{2} < s_2+s_3$, the solution u satisfies the a priori estimate

$$||u||_{(s)} \sim ||f||_{(s)} + ||u||_{(s-e_2)}.$$

Moreover, since the reduced problem of (5.1.2) has a unique solution, there is the equivalency $||u||_{(s)} \sim ||f||_{(s)}$ uniformly with respect to ε.

<u>Example 5.1.3</u>. Consider the boundary value problem

$$(5.1.3) \quad \begin{cases} \pi\ p(1,-i\frac{\partial}{\partial x})(1-\varepsilon^2\Delta)u(x) = f(x), & x \in U \\ \pi_0 b_1(x',\varepsilon,-i\frac{\partial}{\partial x})u = \phi_1, & x' \in \partial U \end{cases}$$

where $f \in H_{(s-2e_3)}(U)$ (with $e_3 = (0,0,1)$), ϕ_1 are given and where the order of b_1 is $\mu_1 = (\gamma_1,m_1,p_1)$. Assume that $m_1 > -1$ and that the principal part b_{10} of b_1 satisfies the coerciveness condition

$$b_{10}(x',1,iN) \neq 0, \qquad \forall x' \in \partial U$$

$$(5.1.4) \qquad \frac{\rho^{\gamma_1-p_1}}{\langle\rho\rangle-\rho}(\rho(\sqrt{2}-1)b_{10}(x',\rho,\omega'+i\quad N)+(\langle\rho\rangle-\rho\sqrt{2})b_{10}(x',\rho,\omega'+i\frac{\langle\rho\rangle}{\rho}N))$$

$$\neq 0, \quad \forall x' \in \partial U, \forall \rho \in (0,\infty].$$

Then Theorem 4.2.1 implies that for $-\frac{1}{2} < s_2 < m_1+\frac{1}{2}$, $m_1+p_1+\frac{1}{2} < s_2+s_3$, the solution $u \in H_{(s)}(U)$ satisfies (with $\sigma_1 = s-\mu_1-\frac{1}{2}e_2+(s_2-m_1-\frac{1}{2})e$) the following a priori estimate:

$$||u||_{(s)} \sim ||f||_{(s-2e_3)} + [\phi_1]_{(\sigma_1)} + ||u||_{(s-e_2)}.$$

In this case, the reduced problem is given as follows:

$$\pi p(1,-i\frac{\partial}{\partial x})l_0u(x) = f(x), \qquad x \in U.$$

Let now $m_1 < -1$. If b_{10} satisfies (5.1.4) and the condition

$$b_{10}^0(x',\omega'+i \quad N) \neq 0, \qquad \forall x' \in \partial U,\ \forall\omega' \in \Omega_{n-1},$$

then Theorem 4.2.2 implies that for $-\frac{3}{2} < s_2 < -\frac{1}{2}$,
$\max(-1,m_1+p_1) + \frac{1}{2} < s_2+s_3 < \frac{1}{2}$, the solution $u \in \overset{\circ}{H}_{(s)}(U)$ satisfies the following estimate (with $\tau_1 = s-\mu_1 - \frac{1}{2}e_2$):

$$||u||_{(s)} \sim ||f||_{(s-2e_3)} + [\phi_1]_{(\tau_1)} + ||u||_{(s-e_2)}.$$

The reduced problem is given by

$$\pi\ p(1,-i\frac{\partial}{\partial x})u(x) = f(x), \quad x \in U$$

$$\pi_0 b_1^0(x',-i\frac{\partial}{\partial x})u = \phi_1, \ x' \in \partial U$$

where $u \in \overset{\circ}{H}_{s_2}(U)$ is sought.

Example 5.1.4. Consider the boundary value problem

$$\text{(5.1.5)} \qquad \begin{aligned} &\pi(1-\Delta)\frac{1-\varepsilon^2\Delta}{2-\varepsilon^2\Delta}\, u(x) = f(x), && x \in U \\ &\pi_0 b_1(x',\varepsilon,-i\frac{\partial}{\partial x})u = \phi_1, && x' \in \partial U \end{aligned}$$

where, as above, b_1 has order $\mu_1 = (\gamma_1,m_1,p_1)$. Since $r_2-1 = k_0 = 0$, (3.2.10) is violated and Theorem 4.2.2 will be applied. Consider first the case $m_1 > 0$. Then $l_0 = 0$. If b_{10} satisfies the coerciveness condition

$$b_{10}(x',1,iN) \neq 0, \qquad \forall x' \in \partial U$$

$$\text{(5.1.6)} \quad \frac{\rho^{\gamma_1-p_1}}{\langle\rho\rangle-\rho}((\sqrt{2+\rho^2}-\rho \quad)b_{10}(x',\rho,\omega'+i \quad N)+(\langle\rho\rangle-\sqrt{2+\rho^2})b_{10}(x',\rho,\omega'+ i\frac{\langle\rho\rangle}{\rho}N)) \neq 0, \quad \forall x' \in \partial U,\ \forall\rho \in (0,\infty],\ \forall\omega' \in \Omega_{n-1},$$

Theorem 4.2.2 implies that for $\frac{1}{2} < s_2 < \frac{3}{2}$, $\max(-1,m_1+p_1)+\frac{1}{2} < s_2+s_3 < \frac{1}{2}$ the solution $u \in \overset{\circ}{H}_{(s)}(U)$ satisfies the estimate (with $\sigma_1 = s-\mu_1-\frac{1}{2}e_2+(s_2-m_1-\frac{1}{2})e$):

$$||u||_{(s)} \sim ||f||_{(s-2e_2)} + [\phi]_{(\sigma_1)} + ||u||_{(s-e_2)}.$$

The reduced problem is given by

$$\pi(1-\Delta)u(x) = f(x), \; x \in U$$

where the solution u is sought in $\mathring{H}_{s_2}(U)$. (Note that in the formulation of the reduced problem the boundary condition $\pi_0 b_1 u = \phi_1$ is replaced with the condition $u \in \mathring{H}_{s_2}(U)$, which implies that $\pi_0 u = 0$.)

Let now $m_1 < 0$. Then $l_0 = 1$ and the coerciveness condition can be formulated as follows: (5.1.6) holds, and, moreover, one has

$$b_{10}^0(x',\omega'+i \quad N) \neq 0, \quad \forall x' \in \partial U.$$

Under this assumption, Theorem 4.2.2 implies that for $-\frac{1}{2} < s_2 < \frac{1}{2}$, $\max(-1,m_1+p_1)+\frac{1}{2} < s_2 < \frac{1}{2}$, the solution $u \in \mathring{H}_{(s)}(U)$ satisfies the estimate (with $\tau = s-\mu_1-\frac{1}{2}e_2$):

$$||u||_{(s)} \sim ||f||_{(s-2e_2)} + [\phi_1]_{(\tau_1)} + ||u||_{(s-e_2)}.$$

The reduced problem is given by

$$\pi(1-\Delta)u(x) = f(x), \quad x \in U$$
$$\pi_0 b_1^0(x',-i\frac{\partial}{\partial x})u = \phi_1, \; x' \in \partial U$$

where $u \in \mathring{H}_{s_2}(U)$ is sought.

5.2. Reduction to regular perturbations

In this section, it will be shown how several singularly perturbed coercive boundary value problems can be reduced to regular perturbations. Except for the treatment of the problem $\mathfrak{A}_3^\varepsilon$ below, the whole material in this section is taken from [Fr-W].

Let U, l_0, π, π_0 be as in section 4.2. With f, ϕ_1, ϕ_2 given, the following boundary value problem (known as the linear plate problem in elasticity theory) is considered:

$$(5.2.1) \qquad (\varepsilon^2\Delta^2 - \Delta)u_\varepsilon(x) = f(x), \qquad x \in U \subset \mathbb{R}^2$$

$$(5.2.2) \qquad \pi_0 B_j(x', \varepsilon, \frac{\partial}{\partial x})u_\varepsilon(x') = \phi_j(x'), \quad x' \in \partial U, \ j = 1,2$$

where Δ is the Laplace operator and where the boundary operators B_j are respectively as follows:

(i) Clamped plates:

$$(5.2.3) \qquad B_j(x', \varepsilon, \frac{\partial}{\partial x}) = (\frac{\partial}{\partial N})^{j-1}, \quad j = 1,2$$

(ii) Simply supported plates:

$$(5.2.4) \qquad B_j(x', \varepsilon, \frac{\partial}{\partial x}) = (\frac{\partial^2}{\partial N^2} - \sigma\frac{\partial^2}{\partial T^2})^{j-1}, \quad j = 1,2$$

where $\sigma \in [0,1]$ is a given parameter (Poisson ratio) and where $T(x')$ is a unit tangential vector to ∂U at x'.

(iii) Free plates:

$$(5.2.5) \qquad B_1(x', \varepsilon, \frac{\partial}{\partial x}) = \frac{\partial^2}{\partial N^2} - \sigma\frac{\partial^2}{\partial T^2}, \ B_2(x', \varepsilon, \frac{\partial}{\partial x}) = \frac{\partial^3}{\partial N^3} + (1-\sigma)\frac{\partial^3}{\partial T^2 \partial N}.$$

With the boundary value problem (5.2.1), (5.2.2), where B_j are given by (5.2.3), (5.2.4) or (5.2.5), respectively, we associate the Wiener-Hopf operators

$$\mathfrak{A}_1^\varepsilon = \begin{pmatrix} \varepsilon^2\Delta^2 - \Delta \\ \pi_0 \\ \pi_0 \frac{\partial}{\partial N} \end{pmatrix}, \quad \mathfrak{A}_2^\varepsilon = \begin{pmatrix} \varepsilon^2\Delta^2 - \Delta \\ \pi_0 \\ \pi_0 (\frac{\partial^2}{\partial N^2} - \sigma \frac{\partial^2}{\partial T^2}) \end{pmatrix}, \quad \mathfrak{A}_3^\varepsilon = \begin{pmatrix} \varepsilon^2\Delta^2 - \Delta \\ \pi_0 (\frac{\partial^2}{\partial N^2} - \sigma \frac{\partial^2}{\partial T^2}) \\ \pi_0 (\frac{\partial^3}{\partial N^3} + (1-\sigma) \frac{\partial^3}{\partial T^2 \partial N}) \end{pmatrix}$$

The operators which correspond to the reduced problems are:

$$\mathfrak{A}_1^0 = \mathfrak{A}_2^0 = \binom{-\Delta}{\pi_0}, \quad \mathfrak{A}_3^0 = \begin{pmatrix} -\Delta \\ \pi_0 (\frac{\partial^2}{\partial N^2} - \sigma \frac{\partial^2}{\partial T^2}) \end{pmatrix}$$

One can choose the reducing operators for $\mathfrak{A}_1^\varepsilon$ and their quasi-inverses as follows:

$$R_{11}^\varepsilon = \mathrm{Op}\begin{pmatrix} \langle\varepsilon\xi\rangle^2 & 0 \\ 0 & 1 \\ (-i\xi_N + \langle\xi'\rangle)^{-1}, & -|\xi'| \end{pmatrix}, \quad R_{1r}^\varepsilon = \mathrm{Id}$$

$$S_{11}^\varepsilon = \mathrm{Op}\begin{pmatrix} \langle\varepsilon\xi\rangle^{-2} - g_1, & |\xi'| k_1, & k_1 \\ 0 & , 1 & , 0 \end{pmatrix}, \quad S_{1r}^\varepsilon = \mathrm{Id}$$

where

$$g_1(\varepsilon, \xi', \xi_N, \eta_N) = \frac{\langle\varepsilon\xi\rangle + \varepsilon|\xi'|}{i\varepsilon\xi_N + \langle\varepsilon\xi'\rangle} \Pi^-_{\eta_N} \frac{1}{(-i\eta_N + \langle\xi'\rangle)\langle\varepsilon(\xi', \eta_N)\rangle^2}$$

$$k_1(\varepsilon, \xi', \xi_N) = (i\varepsilon\xi_N + \langle\varepsilon\xi'\rangle)^{-1}(\langle\varepsilon\xi'\rangle + \varepsilon|\xi'|).$$

The reducing operators for $\mathfrak{A}_2^\varepsilon$ and their quasi-inverses can be chosen as follows:

$$R_{21}^\varepsilon = \mathrm{Op}\begin{pmatrix} \langle\varepsilon\xi\rangle^2, & 0 \\ 0 & , 1 \\ -1 & , (1+\sigma)|\xi'|^2 \end{pmatrix}, \quad R_{2r}^\varepsilon = \mathrm{Id}$$

$$S_{21}^\varepsilon = \mathrm{Op}\begin{pmatrix} \langle\varepsilon\xi\rangle^{-2} - g_2, & (1+\sigma)|\xi'|^2 k_2, & -k_2 \\ 0 & , 1 & , 0 \end{pmatrix}, \quad S_{2r}^\varepsilon = \mathrm{Id}$$

where

$$g_2(\varepsilon,\xi',\xi_N,\eta_N) = \varepsilon(i\varepsilon\xi_N+\langle\varepsilon\xi'\rangle)^{-1}\Pi^-_{\eta_N}\langle\varepsilon(\xi',\eta_N)\rangle^{-2}$$

$$k_2(\varepsilon,\xi) = \varepsilon(i\varepsilon\xi_N+\langle\varepsilon\xi'\rangle)^{-1}.$$

The solution of (5.2.1), (5.2.2) is sought in the space $H_{(s)}(U)$, where $s \in \mathbb{R}^3$ satisfies the condition $\frac{1}{2} < s_2 < \frac{3}{2} < s_2+s_3$ if B_j are given by (5.2.3) and the condition $\frac{1}{2} < s_2 < \frac{5}{2} < s_2+s_3$ if B_j are given by (5.2.4). Using the operators S^{ε}_{j1}, $j = 1,2$, one can construct asymptotic solutions of the problems (5.2.1), (5.2.2), (5.2.3) or (5.2.4). Let γ be a constant in the interval $(0,\min(1,\frac{3}{2}-s_2))$ or $(0,\min(1,\frac{5}{2}-s_2))$ if B_j are given by (5.2.3) or (5.2.4), respectively. Let the functions $u^{(k)}_{\varepsilon}$ be determined by

$$(5.2.5) \qquad \mathfrak{A}^0_j u^{(k)}_{\varepsilon} = \varepsilon^{-\gamma k} S^{\varepsilon}_{j1}((f,\phi_1,\phi_2)^T - \sum_{0\le k'\le k-1} \varepsilon^{\gamma k'} \mathfrak{A}^{\varepsilon}_j u^{(k')}_{\varepsilon}).$$

Then $u^{(k)}_{\varepsilon} \in H_{(s)}(U)$ and for all $r \ge 0$, one has:

$$(5.2.6) \qquad ||u_{\varepsilon} - \sum_{0\le k\le r} \varepsilon^{\gamma k}\, u^{(k)}_{\varepsilon}||_{(s)} \le C\varepsilon^{\gamma(r+1)}||(f,\phi_1,\phi_2)||_K$$

where K is the space of the data. In order to be more specific, we are going to write down the problem the solution of which is the zeroth order approximation $u^{(0)}_{\varepsilon}$ to the solution of $\mathfrak{A}^{\varepsilon}_2$. In a neighbourhood of ∂U, introduce the coordinates $(x',d) \in \partial U \times \mathbb{R}_+$ by $|x-x'| = \min_{y'\in\partial U} |x-y'| = d(x,\partial U)$. Moreover, let G_n be the fundamental solution of $(1-\Delta)$ in $\mathbb{R}^n$ which decreases at infinity and define

$$K_n(|x'|,x_n) = -2\frac{\partial}{\partial x_n} G_n(x), \; x = (x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+.$$

Finally, let $\chi \in C^{\infty}_0(\mathbb{R})$ be a cutoff function which is identically one on $[-\tau,\tau]$ and the support of which is contained in the interval $[-2\tau,2\tau]$. The number τ is chosen sufficiently small and independent upon ε.

One has to solve the problem

$$-\Delta u_\varepsilon^{(0)} = g_\varepsilon$$

$$\pi_0 u_\varepsilon^{(0)} = \phi_1$$

where g_ε has the following integral representation in terms of the data:

$$\begin{aligned} g_\varepsilon(x) = {} & \varepsilon^{-2} \int_U G_2(\varepsilon^{-1}|x-y|) f(y)dy - \\ & - \varepsilon^{-3}\chi(d(x,\partial U)) \int_{\partial U} K_2(\varepsilon^{-1}|x'-y'|,\varepsilon^{-1}d) \int_U G_2(\varepsilon^{-1}|y'-z|) f(z)dz\, d\sigma_{y'} - \\ & - \varepsilon^{-1}\chi(d(x,\partial U))(1+\sigma)\frac{\partial^2}{\partial T^2} \int_{\partial U} K_2(\varepsilon^{-1}|x'-y'|,\varepsilon^{-1}d)\phi_1(y')d\sigma_{y'} \\ & - \varepsilon^{-1}\chi(d(x,\partial U)) \int_{\partial U} K_2(\varepsilon^{-1}|x'-y'|,\varepsilon^{-1}d)\phi_2(y')d\sigma_{y'}. \end{aligned}$$

The reducing operators for $\mathcal{O}\!\mathcal{L}_3^\varepsilon$ can be chosen as follows:

$$R_1^\varepsilon = \begin{pmatrix} 1-\varepsilon^2\Delta & 0 \\ 0 & \mathrm{Id} \\ \pi_0 t_1 & q_1 \end{pmatrix}, \qquad R_r^\varepsilon = \mathrm{Id}, \qquad S_r^\varepsilon = \mathrm{Id},$$

$$S_1^\varepsilon = \begin{pmatrix} \pi_+(\mathrm{Id}-q_2^{-1}\varepsilon(\,i\varepsilon\xi_N+\langle\varepsilon\xi'\rangle)^{-1}\pi_0 t_1)\mathrm{Op}\langle\varepsilon\xi'\rangle^{-2} 1_0, & -q_1 k_1, & k_1 \\ 0 & , \mathrm{Id} & , 0 \end{pmatrix}$$

where

$$t_1 = -i\xi_N - \frac{\sigma}{(1+\sigma)}\langle\xi'\rangle + \sigma\frac{\langle\xi'\rangle^2}{-i\xi_N+\langle\xi'\rangle}$$

$$q_1 = -\frac{\sigma}{1+\sigma}\langle\xi'\rangle$$

$$q_2 = \varepsilon^{-1}\langle\varepsilon\xi'\rangle - \frac{\sigma}{1+\sigma}\langle\xi'\rangle + \sigma\varepsilon\frac{\langle\xi'\rangle^2}{\langle\varepsilon\xi'\rangle+\varepsilon\langle\xi'\rangle}$$

$$k_1 = q_2^{-1}(i\varepsilon\xi_N+\langle\varepsilon\xi'\rangle)^{-1}\varepsilon.$$

Let $a_j(x)$, $1 \le j \le n$, $a(x)$, be in $C^\infty(\overline{U})$ and let $A_1 = \sum_{1\le j\le n} a_j(x)\frac{\partial}{\partial x_j} + a(x)$. The singular perturbations

$$\mathcal{O}\!\mathcal{L}_{j+3}^\varepsilon = \mathcal{O}\!\mathcal{L}_j^\varepsilon + \begin{pmatrix} A_1 \\ 0 \\ 0 \end{pmatrix}, \qquad 1 \le j \le 3,$$

where the reduced differential operators are not necessarily positive, can be treated with the same reducing operators:

$$R^{\varepsilon}_{j+3,l} = R^{\varepsilon}_{j,l}, \qquad S^{\varepsilon}_{j+3,l} = S^{\varepsilon}_{j,l}, \qquad 1 \le j \le 3$$

$$R^{\varepsilon}_{j+3,r} = R^{\varepsilon}_{j,r}, \qquad S^{\varepsilon}_{j+3,r} = S^{\varepsilon}_{j,r}, \qquad 1 \le j \le 3$$

since $\mathfrak{A}^{\varepsilon}_{j+3}$ and $\mathfrak{A}^{\varepsilon}_{j}$ have the same principal symbol for $1 \le j \le 3$.

The singular perturbation

$$\mathfrak{A}^{\varepsilon}_{7} = \begin{pmatrix} 1-\varepsilon^2\Delta \\ \pi_0 B(x',i\frac{\partial}{\partial x}) \end{pmatrix},$$

where the principal symbol $b_0(x',\xi',\xi_N)$ of B is supposed to satisfy the coerciveness condition

$$b_0(x',0,i) \ne 0 \qquad \forall x' \in \partial U$$

$$b_0(x',\omega',i\rho) \ne 0 \qquad \forall x' \in \partial U,\ \forall \omega' \in \mathbb{R}^{n-1},\ |\omega'| = 1,\ \forall \rho \ge 1$$

has the following quasi-inverse operator:

$$S^{\varepsilon}_{7} = Op(<\varepsilon\xi>^{-2}-g_7(x',\varepsilon,\xi',\xi_N,\eta_N),\varepsilon(i\varepsilon\xi_N+<\varepsilon\xi'>)^{-1}(b_0(x',i\varepsilon^{-1}<\varepsilon\xi'>))^{-1})$$

where

$$g_7(x',\varepsilon,\xi',\xi_N,\eta_N) = \frac{\varepsilon}{(i\varepsilon\xi_N+<\varepsilon\xi'>)b_0(x',\xi',i\varepsilon^{-1}<\varepsilon\xi'>)}\Pi^-_{\eta_N}\frac{b_0(x',\xi',\eta_N)}{<\varepsilon(\xi',\eta_N)>^2}.$$

Let $l(x')$ be a smooth vector field on ∂U such that $|l(x')| = 1\ \forall x' \in \partial U$ and:

$$l(x') = l_N N(x') + l_T$$

where l_N is the normal component of $l(x')$ and l_T denotes the projection of l on the tangential hyperplane to ∂U at x'. For a smooth function $a(x')$, which is strictly positive on ∂U, consider the boundary value problem associated with the operator

$$\mathfrak{A}_8^\varepsilon = \begin{pmatrix} -\Delta \\ \pi_0(-\varepsilon a(x')\frac{\partial}{\partial l} + 1) \end{pmatrix}.$$

If $U \subset \mathbb{R}^n$ and $n = 2$, then this problem has an index which, in general, is different for $\varepsilon > 0$ and $\varepsilon = 0$. However, if n is arbitrary and l satisfies the coerciveness condition

$$l_N(x') > 0, \qquad x' \in \partial U,$$

the singular perturbation $\mathfrak{A}_8^\varepsilon$ can be reduced to a regular one, using the following operators:

$$R_{81}^\varepsilon = \mathrm{Op}\begin{pmatrix} 1 & , 0 \\ \varepsilon a(x')(-i\xi_N+\langle\xi'\rangle)^{-1}, & \varepsilon a(x')(l_N|\xi'|-il_T\xi')+1 \end{pmatrix}$$

$$S_{81}^\varepsilon = \mathrm{Op}\begin{pmatrix} 1 & , 0 \\ \varepsilon a(x')(-i\xi_N+\langle\xi'\rangle)^{-1}(\varepsilon a(x')(l_N|\xi'|-il_T.\xi')+1)^{-1}, & (\varepsilon a(x')(l_N|\xi'|- il_T\xi')+1)^{-1} \end{pmatrix}$$

We are going to write S_{81}^ε as a matrix of integral operators in the case $l_N(x') = 1 \;\; \forall x' \in \partial U$. Let T denote the following trace operator:

$$Tf(x') = \int_U F^{-1}_{(\xi',\xi_n)\to(P_{x'}(x'-y'),d(y,\partial U))}(-i\xi_n+\langle\xi'\rangle)^{-1}\chi(d(y,\partial U))f(y)dy$$

where $P_{x'}$ denotes the orthogonal projection on the tangential hyperplane to ∂U in x', y' is defined for y near ∂U by $|y-y'| = d(y,\partial U) = \min_{z'\in\partial U} |y-z'|$ and the cut-off function χ is identically one in a neighbourhood of zero. Moreover, let the function K_n be defined as follows:

$$K_n(x',|z'|) = (2\pi)^{2n}|z'|^{\frac{3-n}{2}} \int_0^\infty \frac{\rho^{\frac{n-1}{2}}}{1+a(x')\rho} J_{\frac{n-3}{2}}(|z'|\rho)d\rho, \quad x \in \partial U, \quad z' \in \mathbb{R}^{n-1}$$

where J_ν denotes the Bessel function of order ν. Then one has:

$$S_{81}^{\varepsilon}\binom{f}{\phi} = \binom{g}{\psi}$$

where $g = f$ and the function ψ has the following integral representation:

$$\psi(x') = -\varepsilon^{2-n} a(x') \int_{\partial U} K_n(x', \frac{|x'-y'|}{\varepsilon}) Tf(y') d\sigma_{y'} + \varepsilon^{1-n} \int_{\partial U} K_n(x', \frac{|x'-y'|}{\varepsilon}) \phi(y') d\sigma_{y'}.$$

In the case $n = 2$, one has:

$$K_2(x', |z'|) = -2 \operatorname{Re}(e^{-i\frac{z'}{a(x')}} (Ci(\frac{|z'|}{a(x')}) + i\, si(\frac{|z'|}{a(x')})))$$

with $Ci(r)$, $si(r)$ defined as follows:

$$Ci(r) = -\int_r^{\infty} \frac{\cos t}{t} dt, \qquad si(r) = -\int_r^{\infty} \frac{\sin t}{t} dt.$$

Finally, consider the following oblique derivative problem in some bounded domain $U \subset \mathbb{R}^2$:

$$\mathfrak{A}_9^{\varepsilon} = \begin{pmatrix} \varepsilon^2 \Delta^2 - \Delta \\ \pi_0 \frac{\partial}{\partial l} \\ \pi_0 \frac{\partial^2}{\partial N^2} \end{pmatrix}$$

where $l = l(x')$ is a smooth vector field which vanishes nowhere on ∂U. Since the coerciveness condition is satisfied, the index of $\mathfrak{A}_9^{\varepsilon}$ does not depend upon $\varepsilon \geq 0$.

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De singulier gestoorde randwaardeproblemen, d.w.z. differentiaalvergelijkingen die door aanwezigheid van een kleine parameter ε aan te duiden, gekenmerkt zijn, verschijnen op natuurlijke wijze in de Mathematische Fysica (Elasticiteitstheorie, Dynamica van vloeistoffen, Diffractietheorie, Enzymische theorie e.z.v.)

De lineaire stabiliteitstheorie (à priori afschattingen die uniform t.a.v. de kleine parameter ε geldig zijn) werd reeds in de jaren zeventig ontwikkeld voor een brede klasse van singulier gestoorde randwaardeproblemen die aan een algebraische voorwaarde van coërciviteit voldoen (zie [Fr 1]). Deze voorwaarde is echter nodig en voldoende opdat à priori afschattingen uniform t.a.v. ε gelden.

Dezelfde coërciviteitsvoorwaarde garandeert dat een singulier gestoord probleem kan worden herleid tot een reguliere storing. Dit laatste vereist echter het gebruik van Wiener-Hopf operatoren met kleine parameter (zie [Fr-W]). Zulke operatoren zonder kleine parameter werden in [B.d.M, V-E] ingevoerd en onderzocht.

In dit proefschrift zijn de in [B.d.M, Fr 1, V-E, V-L] verkregen resultaten tot een ruime klasse van singulier gestoorde Wiener-Hopf operatoren uitgebreid, asymptotische formules voor de oplossingen van de bijbehorende singuliere storingsproblemen zijn aangewezen en hun (asymptotische en gewone) convergentie is bewezen als $\varepsilon \to 0$.

Voor het verkrijgen van deze resultaten moesten niet alleen de in [B.d.M, Fr.1, V-E, V-L] ontwikkelde methoden aan een bredere klasse operatoren aangepast worden, maar ook nieuwe technieken en begrippen ingevoerd worden om de doelstellingen te bereiken. Met name werd b.v. het begrip van singulier gestoorde Wiener-Hopf operatoren met transmissie-eigenschap uniform t.a.v. de kleine parameter in dit proefschrift naar voren gebracht en speelt een belangrijke rol in de formulering van het stabiliteitsresultaat voor deze operatoren.

De in [Fr-W] ingevoerde en op het lineaire platen-probleem in de Elasticiteitstheorie toegepaste factorisatiemethode werd in dit proefschrift uitgebreid tot de singulier gestoorde Wiener-Hopf operatoren met rationale symbolen. Dit laatste leidde tot het verkrijgen van nieuwe asymptotische formules voor de oplossingen van de bijbehorende randwaardeproblemen. De factorisatiemethode werd ook een geschikt hulpmiddel om de stabiliteit van het index voor de bijbehorende elliptische randwaardeproblemen te bewijzen t.a.v. de coërcieve singuliere storingen.

De ingevoerde en onderzochte klasse van singulier gestoorde Wiener-Hopf operatoren lijkt van nut te zijn ook voor andere gebieden binnen de wiskunde (Numerieke - en Functionaal-Analyse) en zal hopelijk later gebruikt kunnen worden voor het onderzoek van quasi-lineaire singulier gestoorde elliptische randwaardeproblemen.

Hieronder volgt een korte schets van de inhoud van het proefschrift. De inleiding is gewijd aan een korte voorgeschiedenis van het probleem en een beknopte beschrijving van verkregen resultaten en ingevoerde technieken. Zij eindigt met een illustratie van de resultaten d.m.v. simpele voorbeelden.

In hoofdstuk 2 zijn de definities van de Sobolev-type ruimten $H_{(s)}$

gegeven en hun later gebruikte fundamentele eigenschappen (imbeddings-, spoorstellingen, e.z.v.) zonder bewijs geformuleerd (zie [Fr 1] voor de bewijzen).

In hoofdstuk 3 zijn families van coërcieve singulier gestoorde Wiener-Hopf operatoren met constante symbolen op een halflijn beschouwd, die van een parameter $\varepsilon \in \mathbb{R}_+$ en parameters $\xi' \in \mathbb{R}^{n-1}$ afhangen. Een tweezijdige à priori afschatting uniform t.a.v. ε en ξ' voor de oplossingen van deze families eendimensionale singuliere storingen en de constructie van reducerende operatoren, die een singuliere storing tot een regulier gestoord probleem herleiden, zijn de centrale resultaten in dit hoofdstuk.

In hoofdstuk 4 zijn de stabiliteitsresultaten en de reductie-procedure uitgebreid tot de coërcieve singulier gestoorde Wiener-Hopf operatoren met variabele symbolen in een begrensd gebied in $\mathbb{R}^n$. De hier en in hoofdstuk 3 beschouwde symbolen zijn rationale functies van de conormale variabele.

Hoofdstuk 5 is gewijd aan de illustratie van de verkregen resultaten d.m.v. concrete voorbeelden en aan hun toepassingen op het Lineaire Platenprobleem in de Elasticiteitstheorie.

Op 13 januari 1956 werd Wolfgang D. Wendt geboren te Bonn. In 1973 behaalde hij het diploma aan het "Ernst-Moritz-Arndt-Gymnasium" te Bonn. Daarna studeerde hij wis- en natuurkunde aan de Universiteit van Bonn. In 1976 werd het "Vordiplom in Mathematik" afgelegd en in 1978 het "Diplom in Mathematik" met als hoofdvak differentiaalvergelijkingen van de mathematische fysica. Vanaf 1978 verricht hij promotieonderzoek (onder leiding van prof.dr. L.S. Frank) en is hij als wetenschappelijk medewerker (assistent van de richting Toegepaste Analyse) verbonden aan het Mathematisch Instituut van de Katholieke Universiteit te Nijmegen. Hij heeft prof.dr. L.S. Frank geassisteerd bij diens onderwijs over bepaalde onderwerpen van de differentiaalvergelijkingen. Bovendien heeft hij onder leiding van prof.dr. L.S. Frank onderzoek uitgevoerd op het gebied van singuliere storingsproblemen. Het proefschrift is gewijd aan een onderwerp uit de Lineaire Singuliere Storingstheorie.

STELLINGEN

1. Het gebruik van coërcieve Wiener-Hopf operatoren met kleine parameter leidt tot nieuwe asymptotische formules en tot nieuwe foutafschattingen voor de oplossingen van coërcieve randwaardeproblemen voor singulier gestoorde differentiaalvergelijkingen (zie [1] waar asymptotische formules en foutafschattingen werden aangegeven voor de oplossing van het Dirichletprobleem voor sterk elliptische, singulier gestoorde differentiaalvergelijkingen; zie [2] waar een algebra van Wiener-Hopf-operatoren zonder kleine parameter werd ingevoerd; zie [3] waar de algebraïsche coërciviteitsvoorwaarde voor singulier gestoorde randwaardeproblemen werd geformuleerd en waar tweezijdige a-priori-afschattingen uniform t.a.v. de kleine parameter zijn bewezen; zie ook [4] waar coërcieve singuliere storingen tot reguliere storingen herleid zijn d.m.v. Wiener-Hopf operatoren met kleine parameter).

[1] M. Vishik, L. Lyusternik, Uspekhi Mat. Nauk 12(5), 1957, pp. 3-122.
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[3] L.S. Frank, Ann. Mat. Pura Appl. 119(4), 1979, pp. 41-113.
[4] L.S. Frank, W.D. Wendt, Comm. PDE 7(5), 1982, pp. 469-535.

2. Voor rationale symbolen met kleine parameter ε geldt de door Vishik en Eskin geformuleerde "smoothness condition" in het algemeen niet uniform t.a.v. ε. Voor de behandeling van coërcieve Wiener-Hopf vergelijkingen met kleine parameter dient daarom de bovengenoemde voorwaarde uitgebreid te worden (vgl. hoofdstuk 3 van dit proefschrift).

3. Zij $\mathfrak{A}^\varepsilon$ een coërcief, singulier gestoord rendwaardeprobleem en zij $\mathfrak{A}^0$ het bijbehorende gereduceerde probleem. Dan is het aantal randvoorwaarden in $\mathfrak{A}^0$ afhankelijk van de graad van de randoperatoren die in $\mathfrak{A}^\varepsilon$ voorkomen (vgl. stelling 3.2.9 van dit proefschrift).

4. Zij $\mathfrak{A}^\varepsilon$ en $\mathfrak{A}^0$ de operatoren die met een coërcief, singulier gestoord randwaardeprobleem en met het bijbehorende gereduceerde probleem corresponderen. Dan kan men operatoren R_l^ε, R_r^ε (de zgn. links- en rechtsreducerende operatoren) construeren, zo dat met een constante $\gamma > 0$ de volgende afschatting geldt:

$$\mathfrak{A}^\varepsilon = R_l^\varepsilon \mathfrak{A}^0 R_r^\varepsilon + O(\varepsilon^\gamma), \quad \varepsilon \downarrow 0$$

(vgl. [4], waar de factorisatie $\mathfrak{A}^\varepsilon = R^\varepsilon \mathfrak{A}^0 + O(\varepsilon^\gamma)$ werd ingevoerd voor coërcieve singulier gestoorde randwaardeproblemen voor differentiaalvergelijkingen).

5. Zij $U \subset \mathbb{R}^n$ een begrensd gebied, $\phi : \partial U \to \mathbb{R}_+$ een functie, $\lambda > 0$ een parameter en Δ de Laplaceoperator. Zij $\chi_+(s)$ de Heavisidefunctie (met $\chi_+(0) = 0$) en u^λ de oplossing van het randwaardeprobleem

$$(1) \quad \begin{cases} -\Delta u^\lambda + \lambda \chi_+(u^\lambda(x)) = 0 & x \in U \\ u^\lambda(x') = \phi(x'), & x' \in \partial U \end{cases}$$

Zij λ_c, de zgn.kritieke waarde, gegeven door
$\lambda_c \overset{\text{def}}{=} \sup\{\lambda \mid u^\lambda(x) > 0 \ \ \forall x \in U\}$. Dan geldt in het geval
$U = \{x \in \mathbb{R}^n \mid |x| < 1\}$ de volgende scherpe afschatting:

$$m_{1-\frac{n}{2}}(\phi) \leq (2n)^{-1}\, \lambda_c(\phi) \leq m_1(\phi) \quad \forall\phi$$

waarbij $m_t(\phi) \overset{\text{def}}{=} ((\text{meas}(\partial U))^{-1} \int_{\partial U} \phi^t)^{\frac{1}{t}}$

vgl. [5] L.S. Frank, E.W.C. van Groesen, in: Proceedings of the Conference on Analytical and Numerical Approaches to Asymptotic Problems, North-Holland, 1980

[6] L.S. Frank, W.D. Wendt, C.R.A.S. Paris, t. 294(1982), pp. 741-744

[7] L.S. Frank, W.D. Wendt, te verschijnen in: J. Diff. Eq.

6. Zij $U \subset \mathbb{R}^2$ een enkelvoudig samenhangend, begrensd gebied. Dan heeft de kritieke oplossing $u^{\lambda_c}(x)$ van (1) de volgende eigenschap: de verzameling E_c van de nulpunten van u^{λ_c} is eindig (vgl. [7]).

7. Zij $U \subset \mathbb{R}^2$ een begrensd gebied (niet noodzakelijk enkelvoudig samenhangend). Dan is de verzameling E_c van de nulpunten van u^{λ_c} de vereniging van eindig veel punten en eindig veel gesloten analytische krommen (vgl. [7]).

8. Zij $U \subset \mathbb{R}^n$ een begrensd gebied met voldoende gladde rand ∂U. Zij $E_0(u^\lambda) = \{x \in U \mid u^\lambda(x) = 0\}$ en voor $x \in U$ zij $(x',\rho) \in \partial U \times \mathbb{R}_+$ gegeven door $\rho = \min_{y'\in\partial U} |x-y'| = |x-x'|$. Dan bestaat er een constante C, zo dat $\partial E_0(u^\lambda)$ (de zgn. vrije rand) voor $\lambda >> 1$ een deelverzameling is van

$$S_\lambda = \{x \in U \mid |\sqrt{2\phi(x')\lambda^{-1}} - \rho| \leq C\, \lambda^{-1}\}$$

(vgl. [6,7]).

9. Zij $f(s) = (s+1)^{-1}s_+$ met $s_+ = \max(s,0)$. Voor elk $\varepsilon > 0$ zij u_ε^λ de oplossing van

$$(2)\quad \begin{cases} -\Delta\, u_\varepsilon^\lambda(x) + \lambda\, f(\varepsilon^{-1}\, u_\varepsilon^\lambda(x)) = 0, & x \in U \\ u_\varepsilon^\lambda(x') = \phi(x') > 0, & x' \in \partial U. \end{cases}$$

Onder de aanname dat de in Stelling 8 ingevoerde vrije rand $\partial E_0(u^\lambda)$ een voldoende gladde variëteit is, geldt de volgende afschatting:

$$C^{-1}\varepsilon^{3/4} \le \|u_\varepsilon^\lambda - u^\lambda\|_{H_1(U)} \le C\,\varepsilon^{3/4} \qquad \forall \varepsilon \in (0,1]$$

met een van ε onafhankelijke constante C.

(Vgl. [8] L.S. Frank, W.D. Wendt, C.R.A.S. Paris, t. 295(1982), pp. 451-454.

vgl. ook [9] J.-L. Lions, Lecture Notes in Math. 323, Springer, 1973

[10] C.M. Brauner, B. Nicolaenko, in: Computing methods in Applied Sciences and Engineering, North-Holland, 1980

waar de afschatting $\|u_\varepsilon^\lambda - u^\lambda\|_{H_1(U)} \le C\,\varepsilon^{1/4}$ werd bewezen.)

10. Het beginwaardeprobleem

$$\frac{\partial u^\lambda}{\partial t} - \frac{\partial^2 u^\lambda}{\partial x^2} + \lambda \chi_+(u^\lambda) = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$

$$u^\lambda(x,0) = x_+^2, \quad x \in \mathbb{R}$$

heeft de gelijkvormigheidsoplossing

$$u(x,t) = ((x^2+2t)(a+b\int_0^{x/\sqrt{t}} (\zeta^2+2)^{-2}\exp(-\zeta^2/4)\,d\zeta)-\lambda t)\chi_+(x-\alpha\sqrt{t}).$$

Hierbij is $\alpha \in \mathbb{R}$ de oplossing van

$$(\alpha^2+2)^{-1} - 2\alpha\int_0^\infty (\zeta^2+2)^{-2}\exp((\alpha^2-\zeta^2)/4)\,d\zeta = \lambda^{-1}$$

en de parameters a,b kunnen als functies van α geschreven worden. (Vgl. [11] L.S. Frank, W.D. Wendt, C.R.A.S. Paris, t. 295(1982), pp. 731-734.)

11. Aangezien het onderwijs op de middelbare school in bepaalde vakken niet of nauwelijks als wetenschappelijk beschouwd kan worden, lijkt het beroep van wiskundeleraar te weinig maatschappelijke waardering te hebben.